Dancing braids

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1 Introduction

The project is investigating annular braids as a model for the structured elements of Scottish country dancing (SCD). The aim is twofold: firstly, to explore how well braids can capture meaningful aspects of the dance as experienced on the dance floor; secondly, to deepen understanding of the annular braid group and its relationship to the better known Artin braid group. The project will take us on an adventure through group theory, topology, and SCD.

2 Background & software foundations

After establishing the mathematical definitions for a geometric braid, an Artin braid and an annular braid, the annular, or "Maypole", braid was chosen as a model for dancers' paths as they move through a dance. With the aid of various hand-written scripts in `perl` and `octave`, braids were generated for a number of standard figures that typically occur in Scottish country dances. Pipe cleaners were used to "play" with simple braids and their closures. This exploratory stage provided a feel for the kind of braids that will appear and how we can work with them.

A student licence for Matlab was set up and `braidlab` [12] installed – this library has functionality for generating Artin braids from path data and working with them, with partial support for annular braids. It was noted that `sagemath` also supports Artin braids and has rich functionality for abstract algebra in general[1]. To ease writeup and aid visualisation, an extension to the LaTeX `braids` library to support annular braids was implemented[2] (example in final section).

3 Braid theory

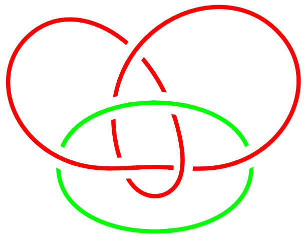
The relationship between the geometric braid group as the mapping class group of an n-punctured disc and the Artin braid group $\mathcal{B}_n$ is well known and explained in the literature, e.g. [10], and has been written up as Chapter 2 of the project, following the Introduction. It is

[1] https://doc.sagemath.org/html/en/reference/groups/sage/groups/braid.html

[2] Submitted as https://github.com/loopspace/braids/issues/8; to date not in the official library.

furthermore known that the annular braid group on n strands $\mathcal{CB}_n$ is isomorphic to the subgroup $\mathcal{D}_{n+1}$ of the Artin braid group on $n+1$ strands [9]: this is the subgroup for which the $(n+1)$th strand begins and ends in the same position (i.e. this strand represents the Maypole). Chapter 3, expected to be complete by the end of April, explores this relationship in more detail.

As part of this chapter, it was necessary to understand how to determine the *closure* of an annular braid: considering the topological model of the punctured disc, it is clear that the core of the cylinder must be treated as a "strand" in the closure, closing to itself (around the outside of the braid).



Left: Annular braid on a cylinder, with a strand in the core; Right: Its closure.

In particular, we describe the isomorphism $f : \mathcal{CB}_n \longrightarrow \mathcal{D}_{n+1}$ (based on [4], [9]) and show that a braid invariant under the equivalence relation, such as *crossing number* (the number of generators in the shortest equivalent braid word), does not generally take the same value for some braid $\alpha \in \mathcal{CB}_n$ as for $f(\alpha) = \beta \in \mathcal{D}_{n+1}$. (Indeed, not all invariants can be calculated directly for the annular braid – this will form the topic of further study in a later chapter).

We have, however, succeeded in demonstrating that an invariant of the *closure* of the annular braid α (i.e., a knot invariant) takes the same value as when calculated for the closure of $f(\alpha)$. Note that this is not the same as saying that the Artin braid based on the line projection of a dance figure d will have the same closure as the annular braid for d – indeed, it is unlikely to! However, as well as outlining the result of [3] that Artin braids generated by projection onto a line at angle are conjugate for different choices of projection angle θ, this theory chapter will establish our result showing how the conjugate Artin braids for a figure do relate to the corresponding annular braid. We also briefly discuss the consequences if we were to "pull the Maypole out" – in topological terms, if the boundary of the punctured disc is not fixed [11].

4 Illustrations: braids and dances

Before diving further into the theory, a short illustrative chapter will follow (the bulk of the work for this chapter is the software foundation, which is complete). First, the braids for a few common "building blocks" of SCD will be shown and discussed briefly – the circle (identity braid), the figures corresponding to the "canonical" (over-under) braid for 3, 4 and 8 dancers, and a progression figure (i.e., the braid has a non-identity permutation). In the second part of the chapter, we briefly analyse the braids for two simple dances. This will give the reader some context for the next chapter, in which we return to the theme of invariants, this time in $\mathcal{CB}_n$.

5 Maypole invariants

Of particular interest to us in this section will be investigating how the typical invariants defined for Artin braids can be calculated for the annular braid. For example, the *complexity* [7] of a braid is a length invariant based on the action of the braid on the n-punctured disc. How should this invariant be calculated for an annular braid? The first part of this May chapter will explore a small number of braid invariants and their annular equivalents – the *complexity*, the *Garside length* [6] of the strands, and the *Lawrence-Krammer* matrix representation [2] (these invariants have been chosen primarily for their mathematical interest, and not their interpretation in SCD).

The second part of the chapter will discuss new invariants specific to the annular braid. We currently have two in mind: (1) first and very simply, an extension to the well known crossing number (mentioned above), namely the net twist of the braid, using the annular-specific "twist" generator τ; (2) the *linking number* of two strands measures how often those two strands change places. Analogously, we will consider interactions between a single strand and the Maypole: how many times does a strand loop around the Maypole? How can this invariant be calculated and how might it relate to other invariants (such as the *winding number* [1])?

6 Braid comparison and orderings of the braid group

An *ordering* in a group G is a relation $<$ on the group elements such that $g < g' \implies hg < hg'$ for all $g, g', h \in G$, i.e. a comparison operator that behaves "nicely" under the group operation; whether such an ordering exists in a given group is in general an interesting mathematical question. It is known that the Artin braid groups permit infinitely many such orderings, but constructing them is nonetheless a delicate matter – testified by the fact that [5] is no conference paper but a 330 page book! Getting to grips with this theory during April will create a solid foundation for the June chapter, which will explore which – if any – braid properties of interest (including of the braid closure) can be used as the basis of such an ordering, i.e. for comparing braids in a way that is meaningful in the context of SCD. In particular, since dancing a sequence with braid β and then dancing it in reverse will yield $\beta\beta^{-1} \cong 1$, we will need to look for properties for which this makes sense – for example, we are unlikely to find an invariant that corresponds to our intuition for *energy*, but we may be able to find one that gives a measure of *left/right balance*. Perhaps the constructive approach to orderings of [5] will also suggest new ways to think about SCD?

7 Quasi-braid models

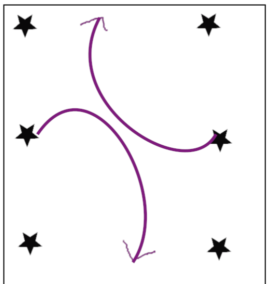
The July chapter of the project will explore how additional structure that exists in SCD might be incorporated into the braid model, and what kind of group or algebraic object we then obtain.

Firstly, since many dances are built from just a small number of figures (e.g. SCD Germany suggests a dozen "basic" figures[3]), we can create a dictionary of braid words for these figures, and consider the subgroup of the braid group generated by just these words. We could also consider how to incorporate constraints based on the bar count, so that eight-bar figures cannot begin "in the middle of" a phrase. Further models that may be considered:

Shoulder strands: by associating a strand with each shoulder of the dancer, we could capture when the dancer turns to face a new direction[4]. How does the constraint on these paired strands (a dancer's shoulders must stay together!) affect the group that we generate? – indeed, we must check that it is still a group. Does it yield new and "interesting" properties?

Strand types: SCD is generally danced in couples, and there is typically a relationship between the tracks of two partners, whether they are moving together, in parallel, mirroring each other, or translated in time. As a consequence, certain braids are more or less likely to be generated by dances. We will consider the subgroups generated by these different symmetries separately, and investigate whether they can provide insights towards a more general model in which partner strands are identified.
(There may be links to the "coloured strand" model of quantum topology [8] in which strands of the same type (spin) are assigned the same colour; the resulting algebraic object is a *groupoid*.)

The problem of the petronella turn: many SCD phrases do not meet the geometric braid requirement that the sets of start and end points are setwise equal. Nonetheless, SCD is certainly not freely choreographed and we can specify a finite grid of points that could be reached by dancers at the end of a phrase. Accordingly, we will consider how the braid model can be extended to include these "potential positions". One approach



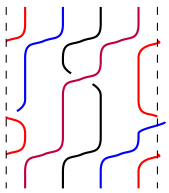
The Petronella turn

[3] https://www.scd-germany.de/cms/upload/pdf/Kurslevel.pdf

[4] Idea thanks to Hugh Griffiths of http://geekknitting.blogspot.com/

would be to use groupoids with a finite (mathematical) set of (dance) set configurations as their objects: we can look at the structure and properties of these groupoids.

This will be the last "maths" chapter of the thesis; a discussion and conclusions chapter will follow but its content is not yet clear.



8 Miscellaneous

A poster on "Dancing Braids", showcasing the data collection/processing aspects of this project and software basis, and omitting the group theoretic detail, will be presented at the BCSWomen Lovelace Colloquium on 4 April.

Above: the annular braid for Set And Rotate, a progression figure

References

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