DANCING BRAIDS

A dissertation submitted to Birkbeck, University of London
for the degree of MSc in Mathematics.

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Contents

Abstract

BIRKBECK, UNIVERSITY OF LONDON

ABSTRACT OF DISSERTATION submitted by **Mair Allen-Williams** and entitled **Dancing braids**.

Date of Submission: 30 August 2024

This project investigates the use of the annular braid for modelling Scottish Country Dancing. SCD dances are written for a fixed number of dancers and typically have a framework of simple figures associated with the home positions of a "dance set". Topological braids are used to model the dancers' trajectories, abstracting away much of the irrelevant detail. In light of the shape of a dance set, the project focuses on the annular, or "Maypole", braid, and explores various features of these and their relationship to the Artin braid. Braids model interactions between strands, thus this approach captures information about how dancers interact with each other; use of the Maypole braid also makes it possible to explore interactions with the Maypole (rotations around the centre of the dance set). However, the equivalence relation associated with braid isotopy creates does not correspond well to a dancer's experience, where the flow and sequencing of the dance figures are key to the feel of the dance. For the same reason, although the visual structure of a braid plotted from trajectory data does make certain symmetries evident, these are not captured in the braid algebra. After a brief detour into the idea of devising dances such that their braid closure has some interesting property, the project concludes by briefly introducing groupoids as an alternative model with potential both to capture the flow of a dance and to explore local symmetries.

Declaration

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In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain.

— Hermann Weyl

1 — Introduction

Scottish Country Dancing (SCD) is a popular dance style in the UK and indeed around the world. It is beloved partly for the strong framework that underpins every dance: a temporal and geometric structure that gives the dance its shape and flow. The temporal dimension of this scaffolding is established by the solid musical framework – eight-bar phrases with a foot-tapping regular beat guide dancers through the dance. The other dimension is buttressed by the form of the **dance set** (fig. 2.1): every dance starts and ends in a fixed configuration of "home positions" called a set; the gravitational pull of this shape can be felt throughout the ebb and flow of the dance. Within this musical and geometric scaffolding, dances are built from a limited collection of standard figures. Yet SCD is also a source of endless challenge – even the most skilled dancers can find new dances to explore, in which the interest lies precisely in the kaleidoscope of patterns that are formed and reshaped within the constraints of this framework. It is thus perhaps not surprising that many technical and mathematical minds are to be found at SCD clubs and events – SCD clearly has a strong mathematical structure.

This project explores the use of a mathematical abstraction to expose the bones of this structure and so to seek insights about SCD and about mathematics. The model we will use is the **braid**[1] – an object that is rooted in topology and which turns out to have interesting group-theoretic properties. In Chapter 2, we introduce the braid as a topological and group-theoretic object, and identify the annular, or "Maypole", braid as a suitable model for SCD. The annular braid is less well known than its older sister, the Artin braid, and so we take some time in Chapter 3 to explore its relationship to the Artin braid. Chapter 4 illustrates the model with a selection of dance figures and dances, before we continue in Chapter 5 to explore some of the properties of braids, what they can tell us about a dance, and also the deficiencies of the braid model. In Chapter 6, we offer a taster of two future directions for using related models to analyse and potentially devise dances: firstly, looking at braid closures and thinking about devising dances that have certain properties; secondly, extending the braid group model into a groupoid.

[1]Credit is due to Roger Picken (Instituto Superior Técnico, Lisbon) for the initial suggestion that braids could be used to model SCD dances, and several interesting conversations during the project.

2 — Braids: definitions

2.1 SCD terminology

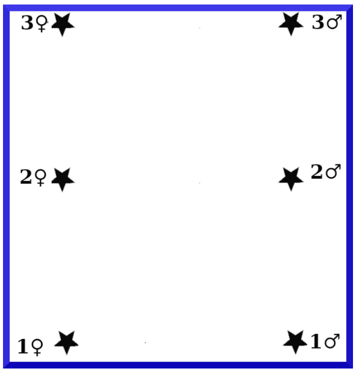


Fig. 2.1: A dance set

Before diving into the mathematical theory, we will need some basic SCD terminology for talking about dances. To dance an SCD dance, pairs of dancers, or **couples**, form into a **dance set**; dance sets have a specified *shape* such as "3 couples longwise" or "4 couples square". The former, shown in fig. 2.1, is the most common and we will typically use it for our examples, but we will see examples of three differently shaped sets in the dances in Ch. 4. One partner in each couple dances the lady's role and one the man's role; we will talk of the "ladies" and the "men". A dance consists of a sequence of **dance figures**, which are defined paths for some or all of the dancers in the set, where the degree of precision is the positions that the dancers should be in at the end of each musical bar. In addition, when moving dancers pass one another in a figure, the instructions will specify whether they pass **by the right** (their right shoulders are nearest) or **by the left**. The instructions will also specify when (and how) dancers should take hands, and may also include movements on the spot. We will be neglecting these latter two points for this project, and focusing on the dancers' trajectories and the fact that these are specified only to the granularity of once per bar. The rest of this chapter introduces the mathematical definitions we will need for our abstraction of these dance paths.

2.2 Geometric braids

A **geometric braid** can be pictured as a collection of n strands suspended between two discs subject to three conditions on the strands, given in fig. 2.2b. We can also consider the strands as a set of n arcs $f_j : [0, 1] \to \mathbb{R}^2$:

$$\{f_j(z_i) | j = 1, \dots n\} = \{(x_i, y_i)\} \qquad \text{(Exactly } n \text{ distinct elements: braid conditions 1 \& 2)}$$

$$\{f_j(0)\} = \{f_j(1)\} \qquad \text{(Setwise equality of start and end points: braid condition 3)}$$

(a) Model of a geometric braid

A geometric braid meets the following conditions on its strands:

1. No doubling back
2. No collisions
3. The set of start positions on the disc is equal (setwise) to the set of end positions

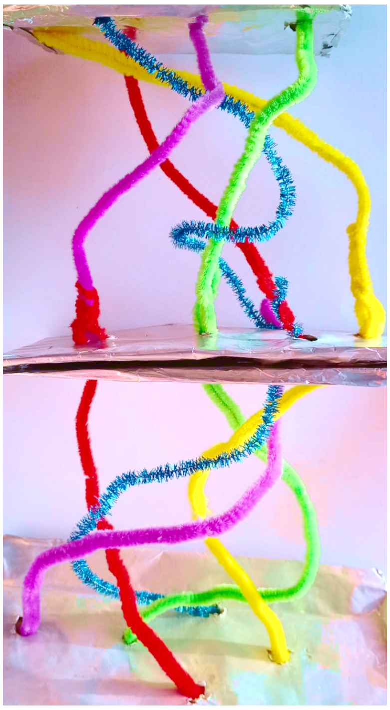
(b) Geometric braid conditions

Fig. 2.2: Geometric braids

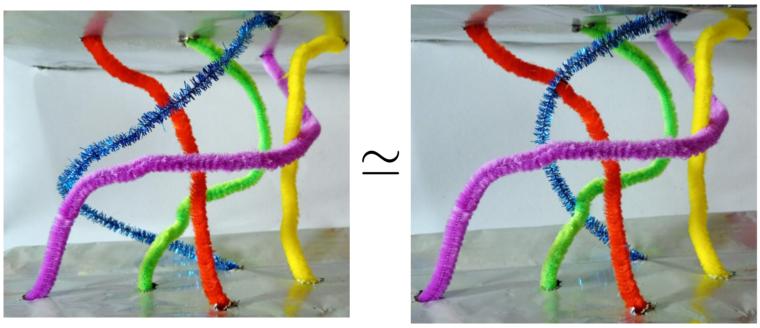
There is a natural mapping from the paths of a set of moving particles (or dancers) onto a geometric braid, taking the z axis to represent time (fig. 2.2a). Typically, we normalise the z axis so that the braid is considered to travel from $z = 0 \to 1$. That this natural mapping meets conditions 1 and 2 is evident: *no doubling back* implies no time travel, and *no collisions* is a physical necessity. Condition 3 allows us to concatenate two braids into a longer braid (fig. 2.3a). Notice that the condition is setwise: we do not require each strand to return to its original starting point. If we label the strands in some way and assign an ordering to them at $z = 0$, e.g. according to their x coordinate, the permutation of this ordering at $z = 1$ is called the **permutation** of the braid. Braids with identity permutation (i.e. each strand ends in its original (x, y) position) are called **pure braids**.

We also define the notions of strand **homotopy** and braid **isotopy**: two strands each starting at the same point s and ending at the same point t are homotopic if it is possible to "pull" or "slide" one of them into the position of the other without affecting any other strands; there is an isotopy between two braids if one can be transformed into the other by moving strands continuously without violating the geometric braid conditions at any stage. Intuitively, this means that if we take our braid, fix the start and end points of each strand, and shake the braid about in any way we can (providing we do not break any strands or introduce any doubling back), the result will be an isotopic braid (fig. 2.3b).

For two braids $\beta_1 = \{s_1, \dots, s_n\}$ and $\beta_2 = \{t_1, \dots, t_n\}$ to be isotopic, there must exist homotopies between n pairs of strands (s_i, t_j). We say that two braids are *equivalent* if there is an isotopy between them and will write $\beta_1 \simeq \beta_2$ or simply $\beta_1 = \beta_2$.



(a) *Concatenation of two braids: we imagine removing the centre plates to create one long braid.*



(b) *Braid isotopy: the blue strand (centre at the top) may make a wide or tight loop around the other strands*

(c) *This braid is isotopic to the identity: all the strands can easily be pulled straight*

(d) *Right: the mirror image of a braid is its inverse*

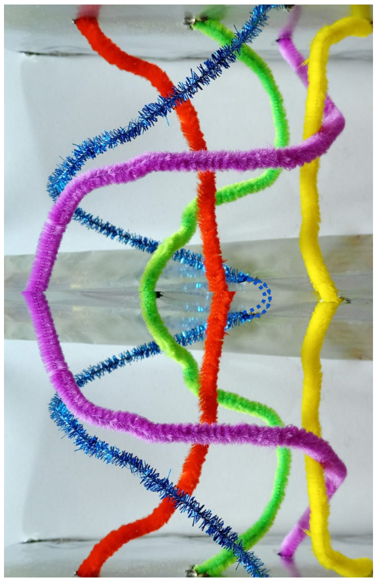


Fig. 2.3: Braid properties: concatenation, isotopy, inverses

2.2.1 The geometric braid group

Using this notion of equivalence (isotopy), we can show that the geometric braids with n strands form a group $\mathcal{GB}_n$, for fixed $n \in \mathbb{N}$. The group operation is concatenation of braids (fig. 2.3a). We fix a set of n start positions S, and note that any braid β with n strands is isotopic to some braid β_S whose start and end positions are S (achieved by extending the strands to the nearest point in S); we can therefore fix S to be a line of points equally spaced along a diameter of a disc, and take $\beta = \beta_S$.

Given this, the requirements for a group are met as follows:

1. **Identity**: We refer to the braid in which all the strands hang straight down from their starting points as the identity braid, which we can write as 1 (fig. 2.3c).
2. **Inverses**: The inverse of a geometric braid β is a braid β' which, when concatenated with β, results in a braid which is isotopic to the identity. We can take β' to be the mirror

image of β about the plane $z = 1$; it is easy to see that $\beta\beta' = 1$ (fig. 2.3d).

3. **Closure**: Let α, β be two braids and $\gamma = \alpha\beta$ their concatenation. Since α, β are braids, γ must satisfy braid conditions 1, 2. Furthermore, since we have the fixed start and end positions S for α, β, we have that the start positions of γ are also S, as are the end positions, and so condition 3 is also met: γ is a braid with n strands.

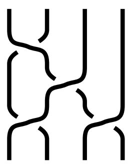
4. **Associativity**: Follows immediately from the definition of concatenation.

The function $\pi : \mathcal{GB}_n \to S_n$ which takes a braid to its permutation is a group homomorphism; its kernel is the group of pure braids $\mathcal{P}_n \lhd \mathcal{GB}_n$.

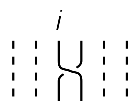
2.3 Algebraic braids: the Artin braid group

We have defined a natural mapping from a dance figure to a geometric braid and shown that these braids form a group. However, these geometric objects can be unwieldy to work with. We introduce a new braid group, the algebraic braid group, which has an efficient notation, and show that it is in fact isomorphic to the geometric braid group.

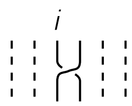
An algebraic braid with n strands is defined by a **braid diagram** depicting (top-to-bottom[1]) crossings of neighbouring strands, which may be *left-over-right* or *right-over-left* crossings. By convention, we define crossings based on their leftmost strand, so a left-over-right crossing is called an **over-crossing** and a right-over-left crossing is an **under-crossing**:



An algebraic braid



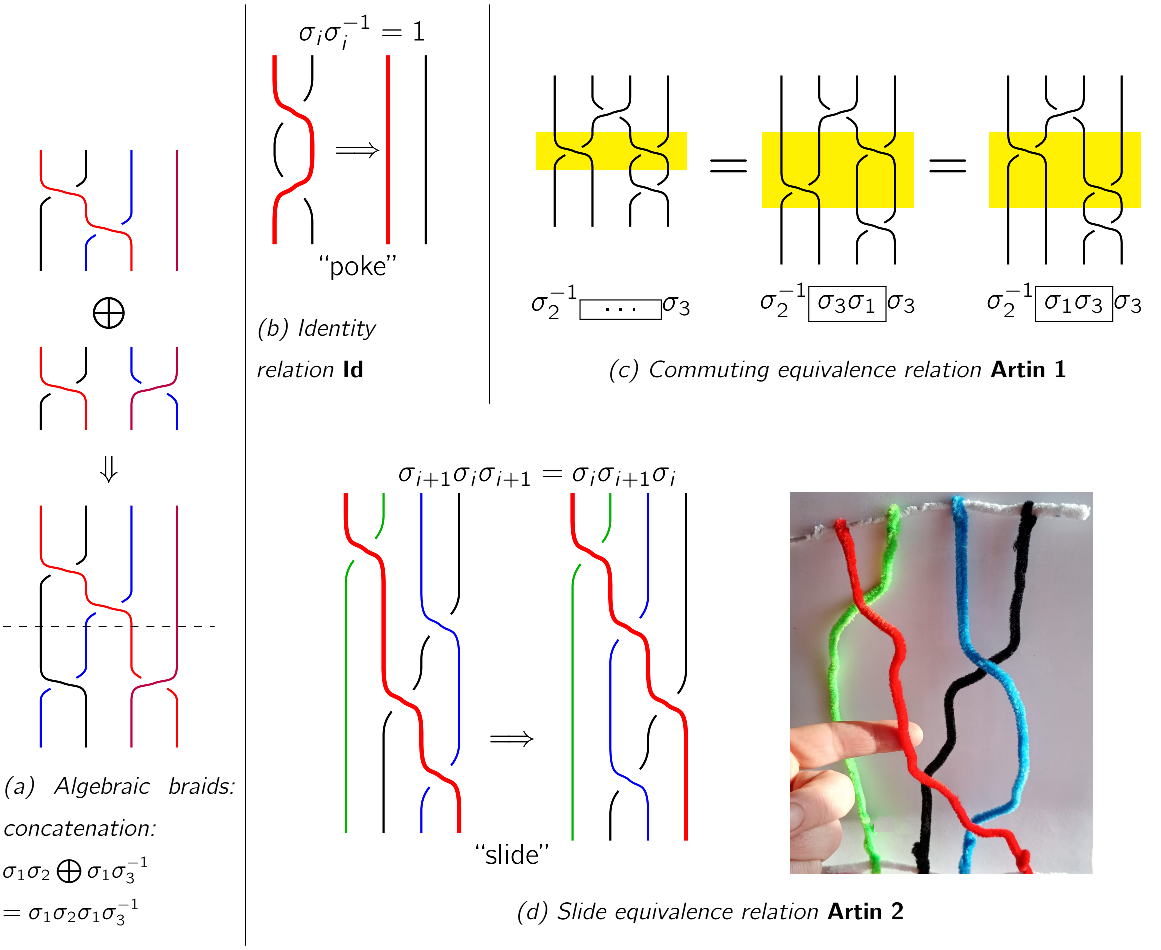
Over-crossing: σ_i



Under-crossing: σ_i^{-1}

A diagram like this is associated with a compact notation, as shown: if strand i (counting from the leftmost strand $= 1$) crosses *over* strand $i+1$ to its right, we write σ_i; if it crosses *under*, we write σ_i^{-1}. If we write down the sequence of σ_i, reading the braid left to right, top to bottom, we obtain a string such as $\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}$, which we call the **braid word**. Just like geometric braids, algebraic braids can be concatenated in the obvious way (fig. 2.4a), and just like the geometric braids, they have an equivalence relation based on isotopy – as before, we can understand this as fixing the ends of the braid, and giving it a shake or a tug to adjust the strands (figs. 2.4b–2.4d). We can write this equivalence relation as relations on the braid word:

[1]We will sometimes draw braids rotated, i.e. left to right; the crossings should be understood *mutatis mutandis*.



(a) Algebraic braids: concatenation: $\sigma_1\sigma_2 \bigoplus \sigma_1\sigma_3^{-1} = \sigma_1\sigma_2\sigma_1\sigma_3^{-1}$

(b) Identity relation **Id**

(c) Commuting equivalence relation **Artin 1**

(d) Slide equivalence relation **Artin 2**

Fig. 2.4: Algebraic braid relations

$$\sigma_i\sigma_i^{-1} = 1 \qquad \textbf{(Id)}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad (|j-i|>1) \qquad \textbf{(Artin 1)}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad \textbf{(Artin 2)}$$

These are sometimes referred to as the **Artin braid relations**, introduced by Artin [Art47], who demonstrates that these relations are sufficient to derive all braid equivalences based on the notion of isotopy as defined above. It is easy to check that the algebraic braids form a group under the equivalence relation, with concatenation as the group operation. This group is called the **Artin braid group**, and we refer to it as $\mathcal{B}_n$. It has been extensively studied (e.g. [Deh21] [MK99] [BB05]) and is known to have a rich group structure. The Artin braid group on n strands is generated by the $n-1$ generators $\sigma_1, \dots, \sigma_{n-1}$, subject to the relations given above. (Note that this is not the only possible presentation: we will discuss another in 5.1).

2.3.1 Geometric $\leftrightarrow$ algebraic braids

We would like to map the geometric braids representing a set of dancer trajectories onto Artin braids, so that we can exploit the power of the algebraic structure and compact notation.

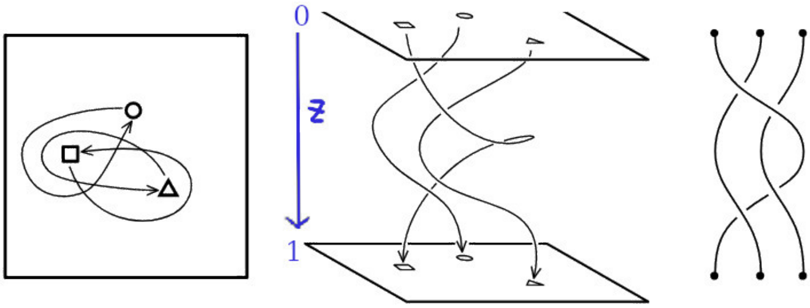


Fig. 2.5: Dancer paths ↔ 3D braid ↔ algebraic braid
Diagram from Roger Picken (Instituto Superior Técnico, Lisbon)[2]

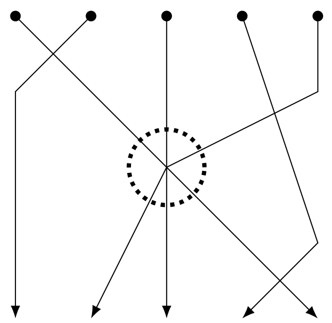
We will find that it is possible to do so without losing any relevant information (i.e., we can guarantee to use the algebraic word to reconstruct a geometric braid that is isotopic to the original geometric braid). The standard approach ([Thi22], [Cao+23], [DE17]) is to project the 3-D geometric object onto a projection line, for example the x axis (fig. 2.5). We can see this as placing an observer outside the area of action. Each time the observer sees two strands change places on our projection line, we must look back at the geometric braid to determine which strand is "in front" from the point of view of this observer and thus whether the crossing is an over-crossing or under-crossing.

We can show that, subject to a little finesse, this mapping is an isomorphism. Full details can be found in standard introductory texts on the mathematics of braids. Here is a brief outline: Let $p_i : \mathcal{GB}_n \to \mathcal{B}_n$ be the projection from a geometric braid $g \in \mathcal{GB}_n$ to a word β in the Artin braid group $\mathcal{B}_n$, using i as the angle of the projection line ($i = 0$ as the x axis; $i = \pi/2$ as the y axis). We need to show that:

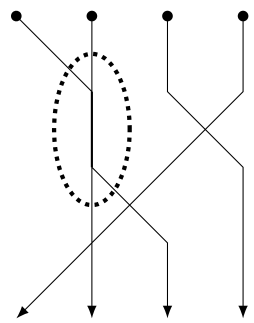
- p_i is a surjection
- p_i is $1:1$
- $p(g_1 g_2) = p(g_1)p(g_2)$, applying the respective group operations (concatenation).

It should be clear that p_i is surjective, since it is easy to picture any braid word as a sequence of crossings of real strings and thus as a geometric braid. Equally, it is easy to see that the concatenation of two geometric braids will be represented by the concatenation of their respective Artin braid words. Showing that p_i is one-to-one requires the "finesse" mentioned above. Specifically, we need to eliminate problems associated with coincident projections. A single point on the z axis at which the projections of exactly two strands coincide corresponds to a crossing in the Artin braid. The problems arise if three or more strands appear to coincide at a single point (fig. 2.6a), or if two strands have coincident projections for a continuous

[2]Source: shared in an email communication



(a) Three points with coincident projections



(b) Coincident projection over a longer period

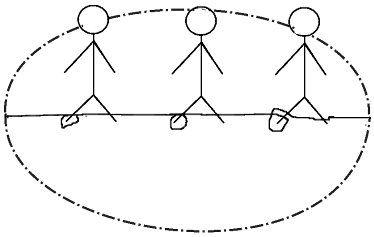
Fig. 2.6: Finessing the isomorphism $p_i : \mathcal{GB} \to \mathcal{B}_n$

sequence $[z_1, z_2]$ of points along the z axis (fig. 2.6b). However, we recall that we are only interested in our geometric braid up to isotopy, and that the geometric braid is a true braid, i.e. none of the strands are physically colliding. We can therefore wiggle the strands a little to obtain an isotopic geometric braid for which the problems do not arise. We refer the reader to the standard braid literature (e.g. [MK99, Ch. 2]) for a more precise analysis. In practice, given a set of trajectories generated from real motion data, rotating the projection line by a fraction of a degree will generally eliminate such coincidences without affecting the algebraic braid. Once such problems have been eliminated it becomes trivial to see that the morphism is $1:1$ and thus an isomorphism.

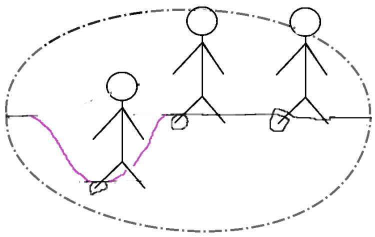
Therefore, $\mathcal{GB}_n \simeq \mathcal{B}_n$ and we can easily justify working with the more convenient notation of the algebraic braid group – we have the certainty that no information in the geometric braid (as a topological object) is lost. We thus have expressions of a braid as a collection of arcs in 3-D space, as a braid diagram, and as a braid word. Next, we look briefly at a very different way of representing a braid.

2.3.2 Curve diagrams

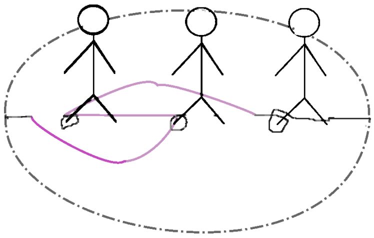
We have modelled a braid as a trajectory through 3-D space. Suppose that, instead, we think of our dancers as moving – still on our flat (2-D) dance floor – in a very viscous fluid, so that the fluid itself is swirled around as the dancers move. As well as looking at the motions of the dancers themselves, we can look at how the fluid is affected by these motions (we continue to work in two dimensions, assuming the particles of the fluid to be moving in parallel with the plane of the dance floor). The formal basis of this model, used in topological dynamics ([Thi22], [Boy94]), is the theory of mapping class groups. Mapping class group theory requires considerable technical background that is out of scope for this project, as our interest is solely in the use of a **curve diagram** to visualise a dance as an action on its surroundings (the hypothetical fluid). For further background on mapping class groups, formal definitions and a technical treatment, we refer the reader to [FM12] or other standard textbooks.



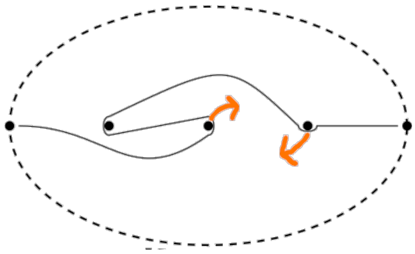
(a) Three dancers as punctures on a disc, with a diameter curve



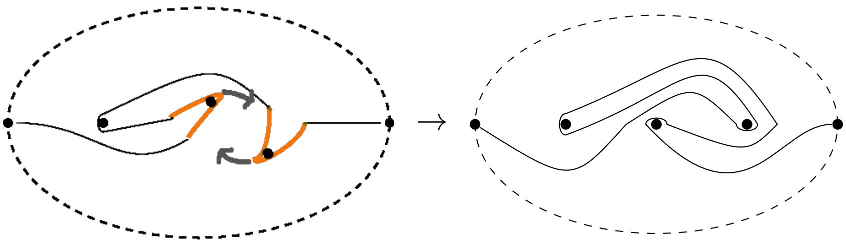
(b) Action on a curve segment as one dancer starts to move



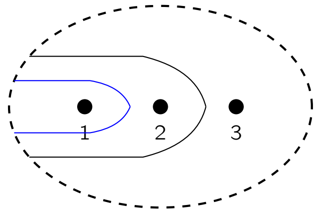
(c) Action on the curve after two dancers have crossed

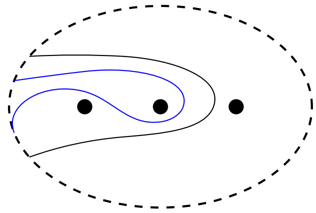


(d) Curve diagram for $\circlearrowleft_{1,2}$; arrows indicate dancers beginning on $\circlearrowright_{2,3}$

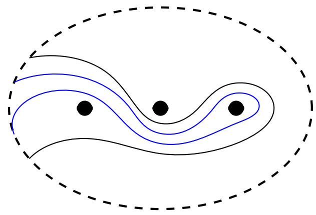


(e) Extending the curve diagram to $\circlearrowleft_{1,2}$;$\circlearrowright_{2,3}$





$\circlearrowleft_{1,2}$



$\circlearrowleft_{1,2}$; $\circlearrowright_{2,3}$

(f) Alternative curve system, action $\circlearrowleft_{1,2}$, action $\circlearrowleft_{1,2}\circlearrowright_{2,3}$

Fig. 2.7: Curve diagram representation of a braid[3]

For our purposes, it is sufficient to know that we can mark a judicially chosen set of curves on the fluid, and map what happens to these as the dancers move. This is done as follows:

As before, we assume that our dancers are moving in a bounded space, and we let this be a disc. Arrange the dancers along a diameter d of the disc, just as we did for a geometric braid. Plot a curve c as a series of arcs along this diameter (fig. 2.7a). We now consider what happens to these arcs as two neighbouring dancers change places by moving around each other (σ_i or σ_i^{-1}). In this model, we picture c as made of elastic, and each dancer to have one foot hooked into this piece of elastic, so that as they travel, the elastic is tugged with them (figs 2.7(b)–(d)). Taking away the dancers, but leaving small holes, or "punctures", in the disc (which prevent the elastic from springing back), and smoothing the curve (isotopy), we have the *curve diagram* of the crossing, presented on the n-punctured disc $\mathbb{D}_n$ (fig. 2.7e).

As the dancers, or punctures, continue to move around each other, they will stretch the elastic

[3]Diagrams drawn with `gimp`, using Wikipedia's stick figure `http://en.wikipedia.org/wiki/Stick_figure`

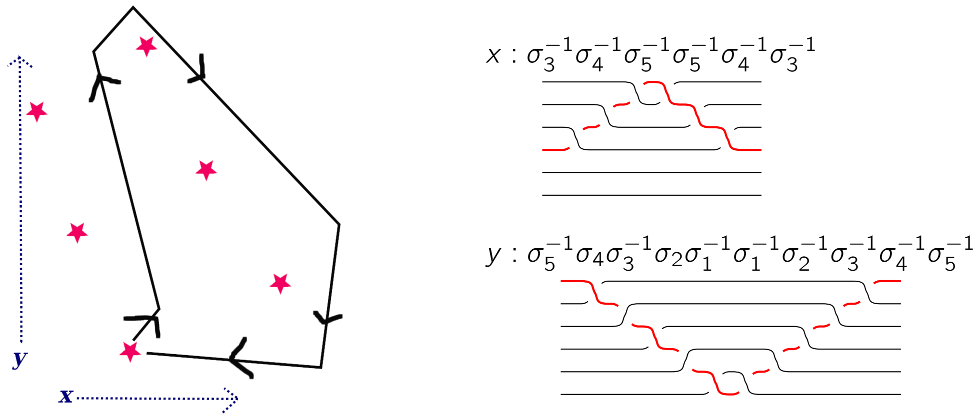


Fig. 2.8: One dancer makes a loop around a line of standing dancers

into a longer and more convoluted curve (unless, of course, they start to perform inverse moves, allowing the elastic to snap back into its original position). The result is the curve diagram for the Artin braid representing the dance. NB: it is important to be clear that these curves on the disc are *not* the arcs representing braid strands. This is an entirely different picture of a braid. Nonetheless, it is possible to prove that, up to isotopy, there is exactly one curve diagram corresponding to each Artin braid (up to equivalence). The curve diagram can be useful for examining certain properties of the braid that are not evident from the braid word or Artin diagram, as we will see in Sec. 5.1.2. We also note that the system of arcs along the diameter is not unique, and other systems can be used (e.g. fig. 2.7f).

Since we have drawn our curve diagram as a sequence of actions corresponding to the crossings in an Artin braid word, a curve diagram shares the property with the Artin braid that it corresponds to a specific choice of projection line for a given geometric braid. Although we have shown in Sec. 2.3.1 that no information is lost in the "translation" from the geometric braid to an Artin braid or curve diagram, the process of projection is not without certain issues.

2.3.3 Problematic projections

Clearly, the isomorphism $\mathcal{GB}_n \simeq \mathcal{B}_n$ is not unique, since different projection lines can result in different Artin braids, as in the example in fig. 2.8. This is not a problem in itself, providing we stick to the same choice of projection line throughout all our braid comparisons. Furthermore, we can see that there must be a relationship between the algebraic braids generated by different projections, since if I dance in front of someone as seen from one angle, and then dance behind them, I must have danced past them as seen from the orthogonal viewpoint. Indeed, it is possible to show that these braids are conjugate. To do this, we will need the notion of closure:

The **closure** of a geometric braid is the object that we obtain by connecting the strand in position x at one end to the strand in position x at the other end, in such a way as to ensure we add no new crossings. If we envisage the braid as living inside a cube or cylinder, the additional

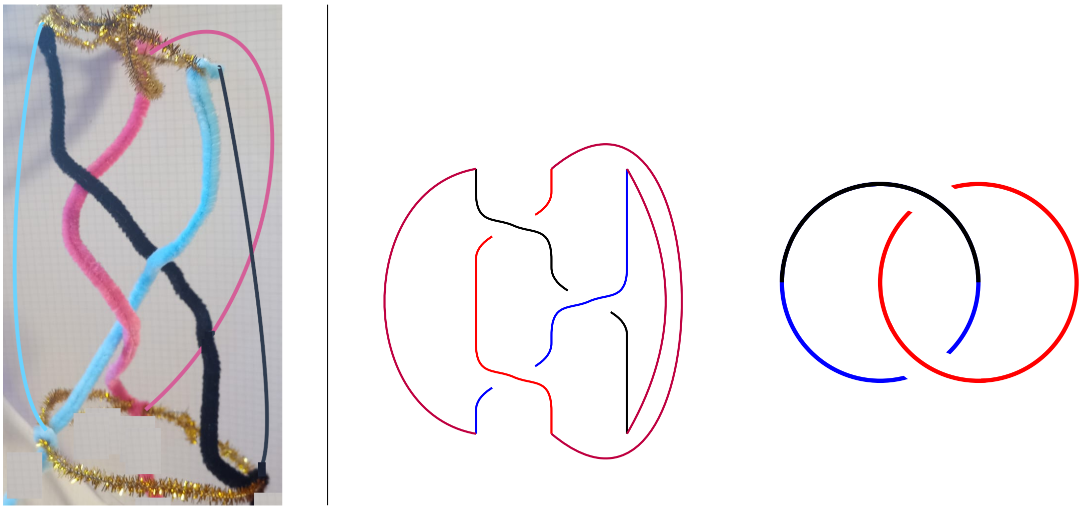


Fig. 2.9: L→R: A geometric braid of $\sigma_1\sigma_2^{-1}\sigma_1$; the closed algebraic braid and an isotopic link

strand sections are outside this volume (fig. 2.9). This object is called a **link** (one or more **knot** components) and, like a geometric braid, we think of it as living in 3-D space. Links, also like braids, are subject to an equivalence relation or isotopy in this space – that is, we can move the strands of a link freely, providing we never cut them or disobey the laws of physics (e.g. by allowing one strand to pass through another), and we will always have an equivalent link.

Given an algebraic braid diagram, we create its closure analogously by drawing strand sections connecting the top and bottom points outside the braid diagram, without introducing additional crossings. The resulting diagram is a projection of a three-dimensional link onto the page. This link is the closure of the geometric braid and so the projection corresponds to the same (isotopic) link regardless of the choice of projection line.

By considering the closure of a geometric braid, we can see that two n-strand algebraic braids will have the same closure:

- If they are the same up to cyclic renumbering (but not reordering) of the strands – i.e. up to rotation of the geometric braid;
- If they are conjugate: suppose that β, α are conjugate braids, with $\beta = \gamma\alpha\gamma^{-1}$. In the closure of β, we can envisage sliding γ^{-1} along the identity section of the strands used to close the braid, so that the link looks like the closure of $\gamma^{-1}\gamma\alpha = \alpha$ (fig. 2.10).

In fact, the second result is stronger: it is possible to show that if two n-strand braids have the same closure, they *must be* conjugate (**Markov's Theorem**: [MK99, Ch. 9]). Now, we stated above that the closure is of the geometric braid and is independent of projection line, so that two algebraic braids corresponding to the same geometric braid must have the same closure. Therefore, by Markov's Theorem, these braids must be conjugate. This is encouraging as it

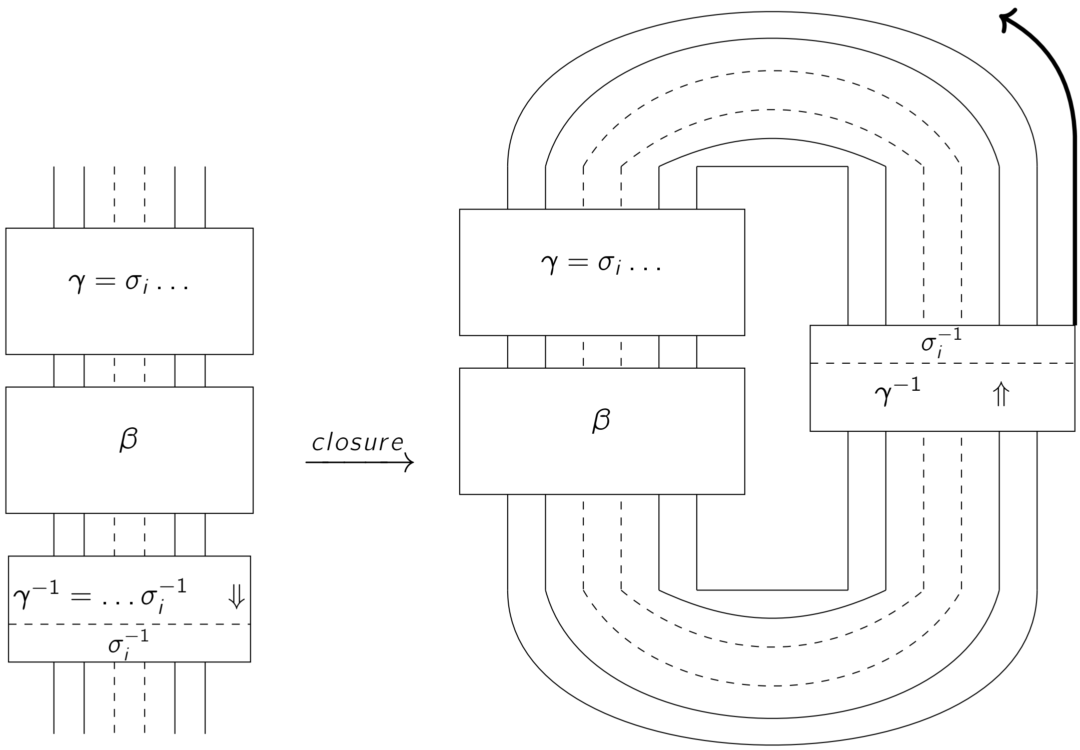
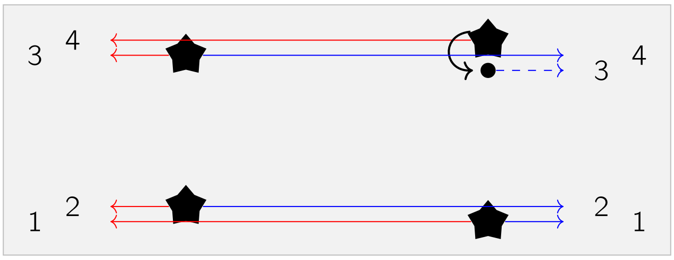


Fig. 2.10: Sliding around a conjugate closure

suggests that even if the braids are not equivalent, there is nonetheless a strong relationship between them. ☐

The second problem with the projection line approach is that it poorly captures the "shape" of a typical Scottish country dance – recall the shape of a typical dance set in fig. 2.1. If we take our projection line to be either the x or the y axis, we will find that we repeatedly face the difficulty of coincident tracks (since the dancers will intentionally be parallel along both axes, at least at the beginning and end of the braid), as well as the problem when using real path data that our braids are very sensitive to slight changes in positioning – see fig. 2.11.

The issue of small movements resulting in renumbering can be fixed by some form of "snap to grid" where we define a fixed set of home positions, and at the beginning and end of the (geometric) braid, artificially move each dancer to the nearest home position. This is necessary in any case for geometric braid condition 3. We can also ease the difficulty by choosing the projection angle to be at $\pi/4$, i.e. at a diagonal to the dance set. However, we still have figures which act "around the set", which a line projection captures rather poorly (for example, the *circle*, fig. 2.13a, or the *grand chain*, fig. 2.13d); there are also many SCD figures that can be danced "on the side" or "across the set", which are intuitively the same for the dancers either way. We would like such figures to have the same braid, at least up to cyclic renumbering of the strands. These issues lead us to consider an alternative to the Artin braid: the annular braid.



The dancers (the four stars) are not standing perfectly in line; we have the strand ordering given by the red arrows $\leftarrow$. Since an SCD dance is organised in couples, ideally we would like all the dancers on the left to be even numbers and on the right odd, or vice versa. More worryingly, if the dancer in the top right adjusts their position just a little bit, the numbering changes (blue arrows $\rightarrow$). Due to geometric braid condition 3, we will always "snap" the dancers to the nearest home position to complete a braid; nonetheless, this sensitivity is a difficulty with the Artin projection.

Fig. 2.11: A difficulty of the Artin projection

2.4 Annular braids

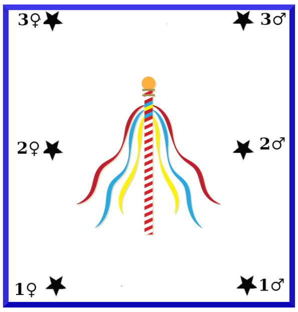
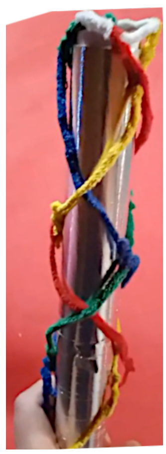


Fig. 2.12

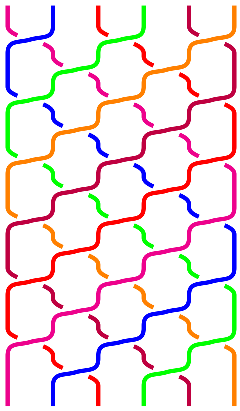
The shape of almost any SCD set is close to a square – or, topologically speaking, a circle. Altering the projection line for an Artin braid is equivalent to moving an observer to a different location outside this circle. However, it may actually seem sensible to position our hypothetical observer at the centre of the set, and project the strands of the geometric braid outward onto a unit circle. The resulting **annular braid** is what we would get if we were to place a Maypole in the centre of the set (fig. 2.12) and clip a Maypole ribbon to each dancer's head, and is thus sometimes referred to as a Maypole braid [Ric09].



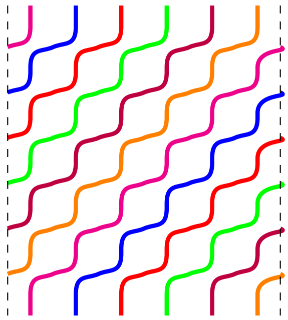
Like the Artin braid, an annular braid is a two-dimensional representation of a three-dimensional object (the geometric braid), and the annular braids with n strands form a group. However, while the Artin braid can be considered as a set of strings hanging down from a line, the annular braid can be considered to be a set of strings hanging down from a circle and must therefore be envisaged as existing on a cylinder (right); we refer to the annular braid group as $\mathcal{CB}_n$.

The annular braid has an algebraic notation similar to the Artin braid, but while the Artin braid on n strands had $n-1$ generators, we now add σ_n, since the nth strand is also able to cross with strand 1 to its right. We also need a new generator for the case where the dancers all simultaneously step to the right or left: this is not observed as a crossing from the Maypole's viewpoint, but we cannot ignore it – for example,

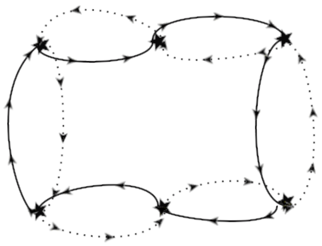
(a) The circle one way[4]



(b) Circle: Artin braid

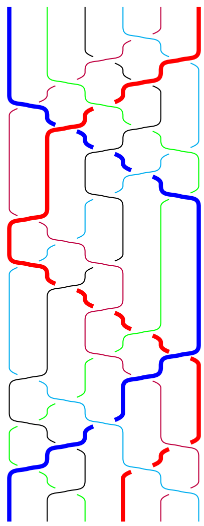


(c) Circle: annular braid

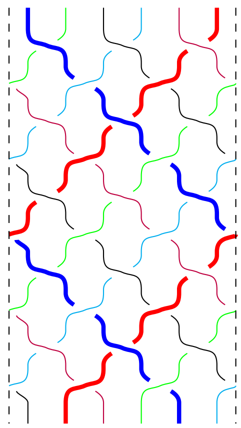


Dancers are labelled alternately 'A' and 'B'; 'A' dancers weave clockwise around the loop and 'B' dancers weave anticlockwise.

(d) The grand chain



(e) Grand chain: Artin braid



(f) Grand chain: annular braid

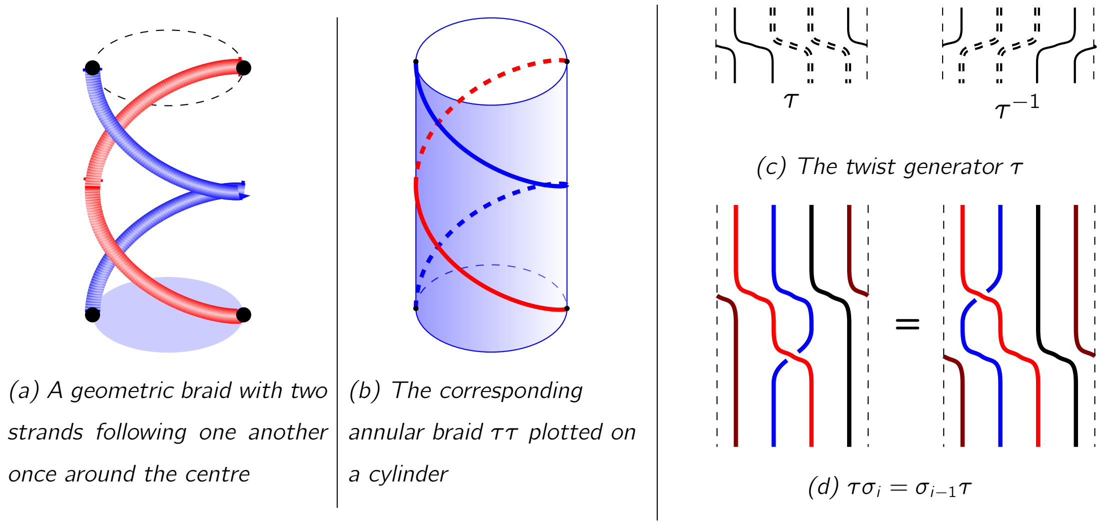
Fig. 2.13: "Round the set" figures and their Artin and annular braids

it can change the overall permutation of the braid. We call this a "twist" and designate the generator τ for a step in the clockwise direction (fig. 2.14c).

We can draw a flat braid diagram on paper by "cutting open" the cylinder along any line parallel to the z axis; the right and left sides of the diagram must be understood to be identified.

The equivalence relations on the crossings are just the same as for the Artin braid group, except that we are now working mod n. Where we see a τ generator next to a sequence of σ generators, we can "slide" the crossing or sequence of crossings along the twist (as in fig. 2.14d) and therefore we have that for any strand i, $\tau\sigma_i = \sigma_{i-1}\tau$.

[4]Image source: `http://www.freepik.com`



(a) *A geometric braid with two strands following one another once around the centre*

(b) *The corresponding annular braid $\tau\tau$ plotted on a cylinder*

(c) *The twist generator τ*

(d) $\tau\sigma_i = \sigma_{i-1}\tau$

Fig. 2.14: The "twist" generator τ

This leads us to the following presentation of the annular braid group $\mathcal{CB}_n$:

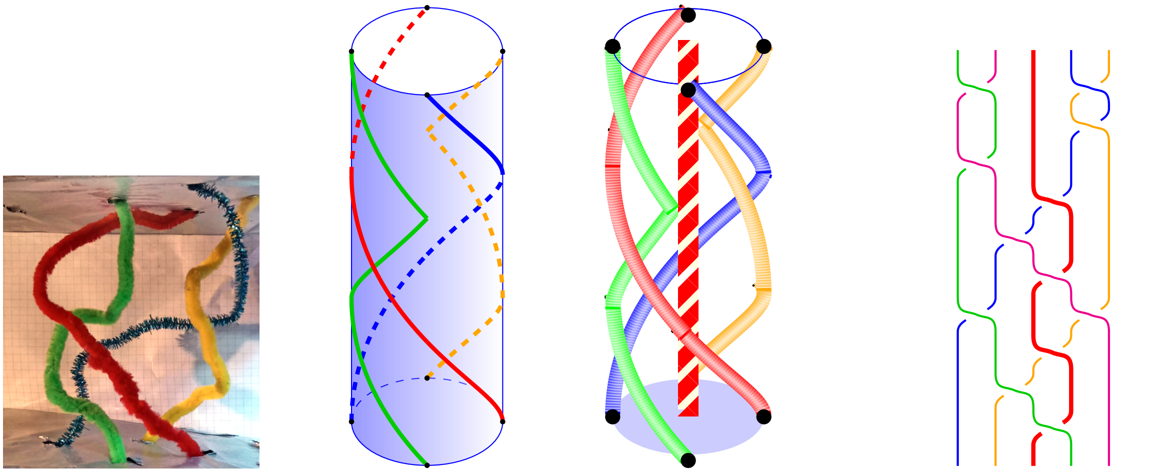
$$\begin{aligned}
\mathcal{P} = \langle\ \sigma_1, \ldots, \sigma_n, \tau \mid\ & \sigma_i\sigma_j = \sigma_j\sigma_i && (|j-i|>1) && (\mathbf{A1}) \\
& \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} && (\text{mod } n) && (\mathbf{A2}) \\
& \tau\sigma_{i+1}\tau^{-1} = \sigma_i\ \rangle && (\text{mod } n) && (\mathbf{T1})
\end{aligned} \tag{2.1}$$

where **A1** and **A2** differ from the equivalent relations **Artin 1** and **Artin 2** only in the fact that all calculations are understood to be mod n.

We can also draw curve diagrams (Sec. 2.3.2) for an annular braid, but we place our dancers on an annulus rather than a disc – that is, the disc has a hole where the "Maypole" is – and the "judicially chosen" system of curves must take this hole into account (we will see an example in Ch. 5).

In the next chapter, we will show that there is an isomorphism between the Maypole braids on n strands and a specific subgroup of the Artin braids on $n+1$ strands, and discuss this relationship further, looking in particular at the Artin and annular braids generated by some dance figure or geometric braid.

3 — Annular braids: a closer look



(a) A geometric braid for SCD figure Set and Rotate[1]

(b) We can project the braid onto a cylinder — annular braid $\tau\sigma_1\sigma_3\tau$

(c) We now contract the cylinder into a strand (of a geometric braid)

(d) We can project this onto an Artin braid $\sigma_1\sigma_4^2\sigma_1\sigma_3\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_2\sigma_3$

Fig. 3.1: Maypole braid as a geometric braid with an extra strand

The core result of this chapter is the isomorphism between the annular braid group and $\mathcal{D}_{n+1}$, the subgroup of the Artin braid group $\mathcal{B}_{n+1}$ for which the $(n+1)$th strand also ends in position $n+1$. To understand this viewpoint geometrically, we can consider the (hypothetical) Maypole to be an additional, straight, strand in the geometric braid (as illustrated in fig. 3.1 – note that different choices of strand numbering / ordering will yield conjugate Artin braids: as with projection lines, the important thing is to make consistent choices).

The algebra in the next section formalises this picture. We will then look more closely at various aspects of the annular braid: its closure; the positioning of the Maypole, and how this compares to choosing the projection line of an Artin braid; how we can understand the extra information that is carried in a Maypole braid (versus the Artin braid for the same set of trajectories, or dance figure); we discuss briefly what happens if we "remove the Maypole", imagining our observer as a hook in the ceiling above the dancers; we also find the centres $Z(\mathcal{B}_n)$ and $Z(\mathcal{CB}_n)$, as these

[1] We will see a description of this figure in Sec. 4.2.4

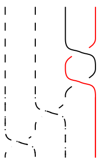
subgroups will prove important to our understanding of the groups. Here and in later chapters, when giving general results where $n = 1, 2, 3$ may be special cases, we assume $n > 3$.

3.1 The Maypole braid group is isomorphic to $\mathcal{D}_{n+1} < \mathcal{B}_{n+1}$

There is an isomorphism between the annular braid group on n strands and the subgroup $\mathcal{D}_{n+1}$ of $\mathcal{B}_{n+1}$ consisting of braids whose permutation always fixes strand $n+1$. We now define this isomorphism. Our approach differs slightly from the presentation of [KP02], [Cho48], but the underlying principles are the same.

Taking the presentation $\mathcal{P}$ (eq. 2.1) of $\mathcal{CB}_n$, we define the isomorphism f as:

$$f : \mathcal{CB}_n \to \mathcal{D}_{n+1} : \begin{cases} f(\sigma_i^k) = \sigma_i & i \in \{1, \dots, n-1\}, k \in \{+1, -1\} \\ f(\tau) = \sigma_n^2 \sigma_{n-1} \dots \sigma_1 & \\ f(\sigma_n^k) = f(\tau^{-1}\sigma_1^k \tau) & k \in \{+1, -1\} \qquad \text{(application of relation } \mathbf{T1}\text{)} \\ f(\beta = \alpha\beta') = f(\alpha)f(\beta') & \alpha \in \{\sigma_i^k, \tau^k\}, k \in \{+1, -1\}, i = 1, \dots, n \end{cases}$$



The key relation here is $f(\tau)$: we can envisage this relation as hooking the last strand "around the Maypole" (σ_n^2) and then across the "back" of the cylinder, while the other strands move up (see left). It follows immediately from the definition that f is well defined and there exists an $(n+1)$-strand Artin braid $\alpha = f(\beta)$ for every $\beta \in \mathcal{CB}_n$. To show that f is an isomorphism, we must show that:

1. $f(\beta) = f(\beta') \iff \beta \simeq \beta'$
2. f is onto: every braid in $\mathcal{D}_{n+1}$ is equivalent to a braid in the image of f

For the first condition:

- Condition 1: $\Leftarrow (\beta \simeq \beta' \implies f(\beta) = f(\beta'))$:
  Observe that we can apply braid relation **T1** to eliminate any σ_n in β, β'. We can therefore assume that if braid relations **A1** or **A2** apply, we can apply **Artin 1** or **Artin 2** to the corresponding elements of $f(\beta), f(\beta')$. Thus we need only consider the relation **T1**. By definition of f, we have that $f(\sigma_n^k) = f(\tau^{-1}\sigma_1^k\tau)$. This leaves the case of $f(\tau^{-1}\sigma_i^k\tau)$, $i > 1$. Fig. 3.2 gives a visual proof that $f(\tau^{-1}\sigma_i^k\tau) = \sigma_{i-1} = f(\sigma_{i-1})$, so the relation holds.
- Condition 1: $\Rightarrow (f(\beta) = f(\beta') \implies \beta \simeq \beta')$:
  We postpone proof of this until after the proof of Condition 2.

For the second condition, we aim to show that every braid in $\mathcal{D}_{n+1}$ is equivalent to a braid which satisfies the following properties:

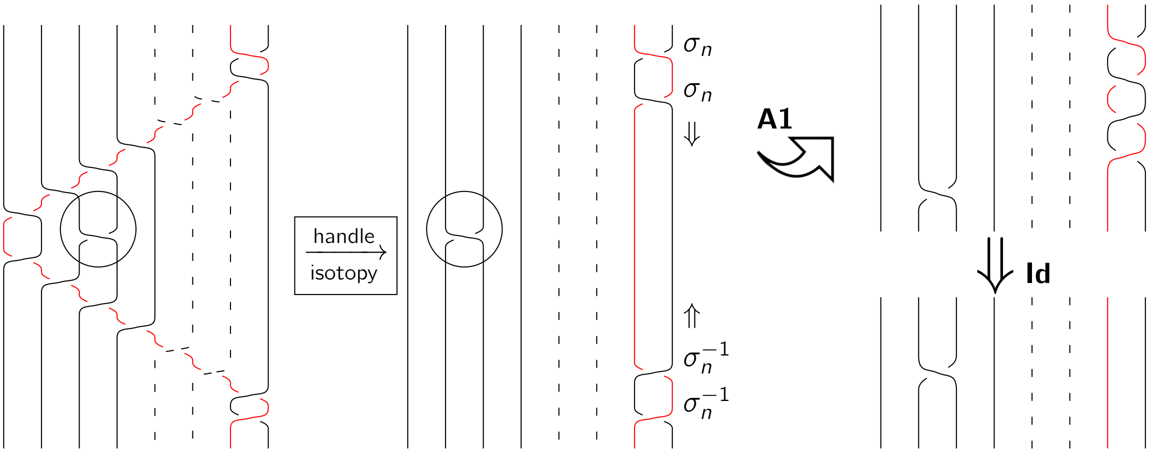


Fig. 3.2: $f(\tau\sigma_i\tau^{-1}) = \sigma_n^2\sigma_{n-1}\ldots\sigma_1\sigma_i\sigma_1^{-1}\ldots\sigma_{n-1}^{-1}\sigma_n^{-2} = \sigma_{i-1}$ $(1 < i < n)$

- Any crossing σ_n^ϵ $(\epsilon = \pm 1)$ is immediately followed by a crossing σ_n^ϵ;
- Any crossing pair σ_n^2 is directly followed by the sequence of crossings $\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1$. We shall call this subbraid $\sigma_n^2\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1 = \gamma$;
- Similarly, any pair σ_n^{-2} is directly preceded by the sequence of crossings $\sigma_1^{-1}\sigma_2^{-1}\ldots\sigma_{n-1}^{-1}$.

Given an $(n+1)$-strand braid $\delta \in \mathcal{D}_{n+1}$ satisfying these properties, we can rewrite it as a word in $\gamma, \gamma^{-1}, \sigma_i$ where $i \in \{1, \ldots n-1\}$. We then have $g = f^{-1}(\delta)$ given directly by:

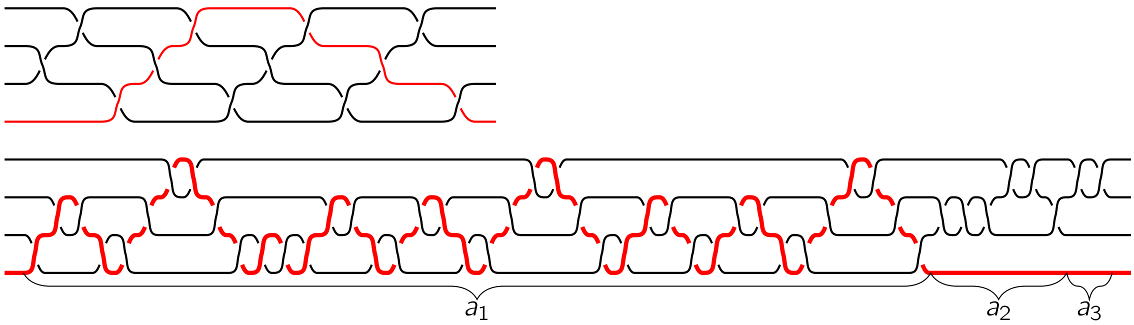
$$g : \mathcal{D}_{n+1} \to \mathcal{CB}_n : \begin{cases} g(\sigma_i^k) = \sigma_i^k & i \in \{1, \ldots, n-1\}, k \in \{+1, -1\} \\ g(\gamma) = \tau & \\ f(\delta = \alpha\delta') = f(\alpha)f(\delta') & \alpha \in \{\sigma_i, \gamma\}, i = 1, \ldots, n-1 \end{cases} \tag{3.1}$$

Notice that it is sufficient to show that we can write every braid $\delta \in \mathcal{D}_{n+1}$ as an equivalent braid meeting just the first condition. We can then achieve the second two conditions by inserting sections equivalent to the identity as necessary (i.e. if $s = \sigma_i\sigma_j\ldots$ is the required sequence, we insert ss^{-1} or $s^{-1}s$). Geometrically, we can see this as pulling strand $n+1$ straight; intuitively, it seems clear that we should always be able to pull *one* strand straight. To show this fact algebraically, we will take a brief detour into Artin's solution to the word problem.

3.1.1 Artin braid combing

An important question for any group is the **word problem**: given two words in the group, are they equivalent under the group relations? In [Art47], Artin gives an algorithm for solving the word problem in $\mathcal{B}_n$ by writing $\alpha = \beta'\beta^{-1}$ in a normal form, such that it is immediately possible to see whether $\alpha \simeq 1$ (and thus $\beta' \simeq \beta$). More efficient approaches to the word problem have since been found (see e.g. [BB05]), but Artin's algorithm for writing α into a normal form remains of interest.

(a) The braid $(\sigma_2^{-1}\sigma_1)^3$ and its combed form $\sigma_1\sigma_2\sigma_2\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1$



(b) The braid $(\sigma_2\sigma_3\sigma_1)^4$ and its (much longer) combed form

Fig. 3.3: Combed forms of the "canonical" braids with 3 and 4 strands

In the word problem, if $\alpha = \beta'\beta^{-1}$ has a non-identity permutation, then we know immediately that $\beta \not\simeq \beta'$. The combing algorithm, designed to solve the word problem, is thus defined for pure braids. Suppose we wish to comb some braid λ that is not a pure braid: let γ be the braid such that $\lambda' = \lambda\gamma$ has identity permutation, where γ is the **permutation braid** given by crossing strand 1 (i.e. the strand whose initial position was 1) behind all strands to the left of it until it is in position 1, then strand 2 similarly, and so recursively. We will comb the pure braid λ' and then compose it with γ^{-1} to obtain a **combed normal form** for $\lambda = \lambda'\gamma^{-1} = \lambda\gamma\gamma^{-1}$.

To comb a pure braid α, we begin from the leftmost strand. Our goal is to use Artin's braid relations to write $\alpha = a_1\alpha_1$ where a_1 is a subbraid in which every crossing involves the strand that was initially in position 1, and α_1 does not involve this strand at all. Since we have arranged that α is a pure braid, strand 1 will be in position 1 at the end of the subbraid a_1, and we now repeat the exercise for α_1 (ignoring strand 1). We continue recursively until we have the braid $\alpha = a_1a_2\ldots a_{n-1}$: after applying **Id**, this is now a **combed braid** – see fig. 3.3.

Artin [Art47] demonstrates that every braid has a combed form (we can always "push down" the crossings not involving the strand of interest at each step), and that it is possible to compute it algorithmically (we can guarantee never to get stuck in a loop[3]). Appendix A gives the derived relations used in the algorithm.

3.1.2 Completing the proof that f is an isomorphism

Notice that in combed normal form, the $(n+1)$th strand never crosses to below position n, and every crossing σ_n^k bringing this strand into position n is immediately followed by another

[3] NB: Care is necessary when choosing how to apply the Artin relations – the author's first attempts at implementing Artin combing programmatically repeatedly got stuck in loops!

σ_n^k. Thus, given a braid $\delta \in \mathcal{D}_{n+1}$, we can find its combed form $\delta_n = \delta$, which will immediately meet the first condition specified above. We insert identity subbraids as described previously to yield $\delta'_n = \delta_n = \delta$ in a form where we can apply $g = f^{-1}$. Therefore, f is onto.

Finally, we return to condition 1: $f(\beta) = f(\beta') \implies \beta \simeq \beta'$. Since a combed braid is in normal form, there is a unique combed braid δ such that $f(\beta) = \delta = f(\beta')$. Therefore, $\beta = f^{-1}(f(\beta)) = f^{-1}(\delta) = f^{-1}(f(\beta')) = \beta'$. This completes the proof that f is an isomorphism. $\square$

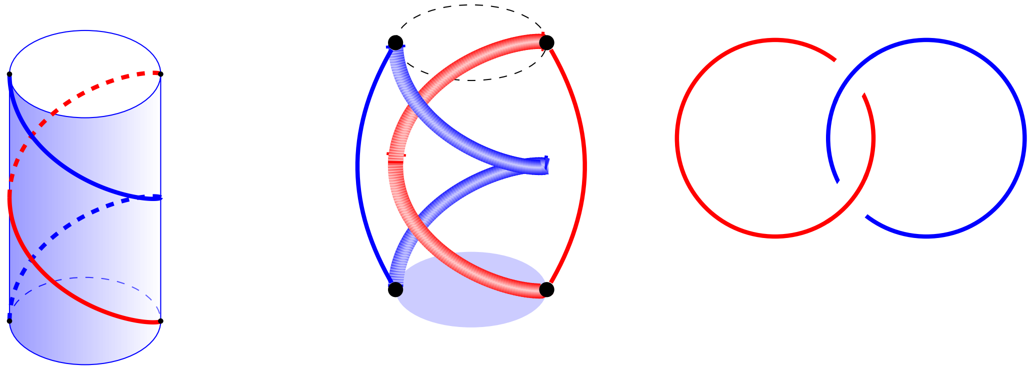
We can now show the following: suppose that δ is an annular braid whose permutation has no 1-cycles, e.g. τ (for $n > 1$); let $\beta \in \mathcal{B}_{n+1} = f(\delta)$ and $\beta' = \sigma_n \beta \sigma_n^{-1}$. Since δ does not fix any strand, we have that β does not fix strand n, thus β' sends strand $(n+1) \xrightarrow{\sigma_n} n \xrightarrow{\beta} s \xrightarrow{\sigma_n^{-1}} s$ where $s \neq n$, and so $\beta' \neq \mathcal{D}_{n+1}$. Hence $\mathcal{D}_{n+1}$ is not a normal subgroup of $\mathcal{B}_{n+1}$. $\square$

3.2 Closure of a Maypole braid

We will also make use of the closures of annular braids. Consider the annular pure 2-braid $\tau\tau$ (fig. 2.14). If we consider this as a geometric braid and connect the ends of each strand (fig. 3.4b) to form the closure, we obtain the simple Hopf link shown in fig. 3.4c, which is (by construction) the same link as we would obtain for the closure of any projection of this braid to an Artin braid. Although there are no crossings in the annular braid (only the τ generators), the link has two crossings. This may seem a reasonable definition for the closure of the annular braid. However, it ignores the additional information carried in an annular braid, namely the position of the annulus, or Maypole, in relation to the strands. An alternative is the approach implied (although not explicitly defined) in `braidlab` [TB19]: to treat the Maypole as a strand in the closure. Fig. 3.6 shows the annular braid $\sigma_1\sigma_2\sigma_1$ and its closure including a Maypole strand at the core of the cylinder. Notice that now, the closure is precisely the closure of the isomorphic $(n+1)$-strand Artin braid (fig. 3.6, right) – this follows from the construction of the isomorphism f. Furthermore, although we mentioned that f is not unique, this closure must be the same under any rotation of f (since it is defined from the geometric braid).

Which closure is "correct"? This depends on the question we hope to answer. In the second case, the closure will always include a distinct link component (an unknot) associated with the annulus, so that, for example, we will never have a closure with a single component. Among other things, this means that while any knot or link can be represented as the closure of some braid[4] (e.g. [MK99]), only certain links (and no knots) can be represented as the closure of an annular braid. Thus if our question is "what braid has this link as its closure?", we may prefer to ignore the Maypole. This also corresponds to our knowledge that there is no physical Maypole or hole in the dance floor. On the other hand, the additional information carried in

[4]indeed, this fact originally motivated the study of braids

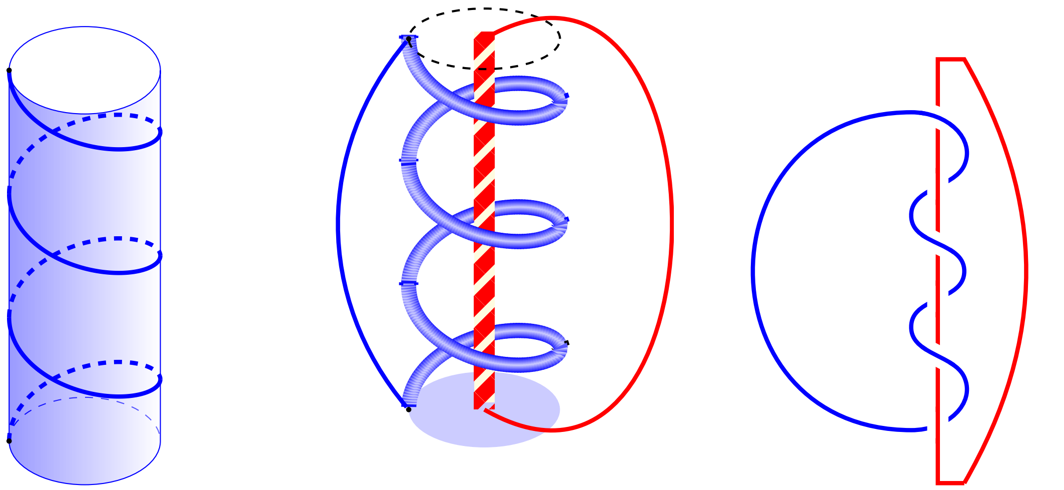


(a) The 2-strand annular braid $\tau\tau$: each strand makes one full rotation

(b) Adding the closure to the associated geometric braid

(c) Artin closure of $\tau\tau$ – the Hopf link

Fig. 3.4: Artin closure of a braid



(a) The 1-strand annular braid $\tau\tau\tau$

(b) Maypole closure of the braid

(c) Link of the Maypole closure

Fig. 3.5: Maypole closure of a braid

the second type of closure may be relevant to other questions. Consider the case of a 1-strand braid corresponding to k loops clockwise around the Maypole (fig. 3.5). Without the Maypole, this braid closes to the unknot. However, if the Maypole is included in the closure, the closure link will carry information about the net number of times the strands pass around the Maypole; for some things, this may be useful. We will be clear when we are working with closures and closure invariants whether we are using the "Artin" or the "Maypole" closure.

3.3 The centres of the braid groups

The centre of a group is often important to understanding the structure of the group. Garside [Gar69] defines the **fundamental braid** for Artin groups as Δ, where Δ^2 is given geometrically as a 360° twist of the identity braid (fig. 3.7). For $\mathcal{B}_n$, Δ^2 can be written as $(\sigma_1 \dots \sigma_{n-1})^n$; for $\mathcal{CB}_n$, we have $\Delta^2 = \tau^n$. It is well known [FM12] that the centre of $\mathcal{B}_n$ is generated by Δ^2. Rather than reiterate the classical proof of this fact, let us consider the annular braid:

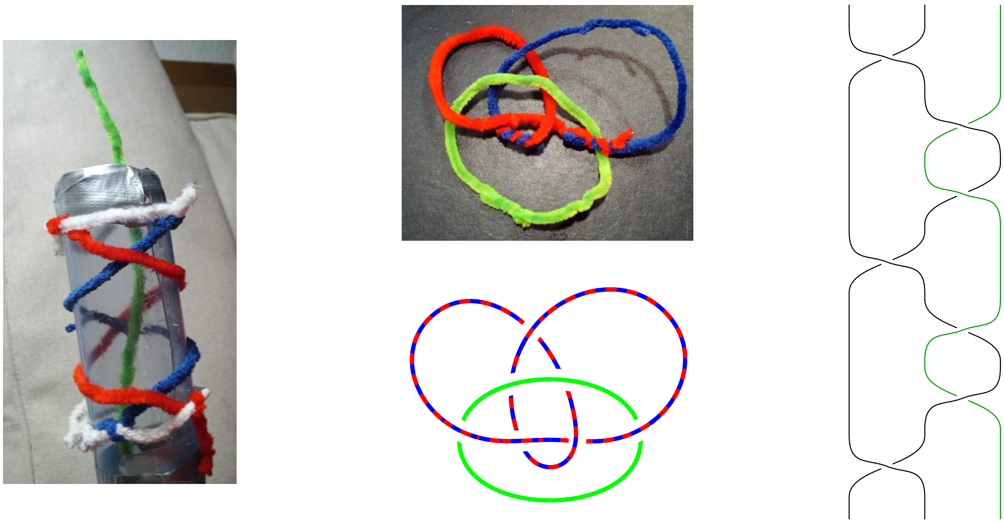
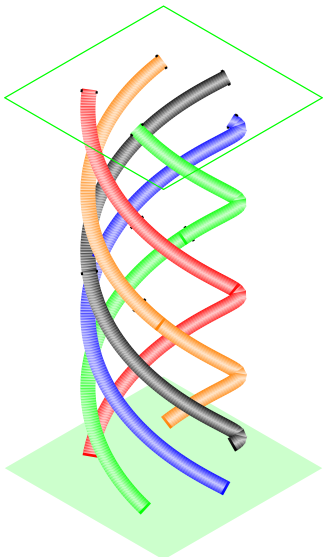
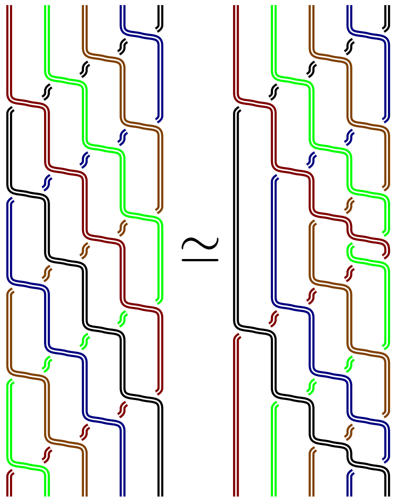


Fig. 3.6: Left: the annular 2-strand braid $\sigma_1\sigma_2\sigma_1$ shown on a cylinder, with a strand (or "Maypole") in the core; Centre: its closure; Right: the isomorphic Artin braid

(a) Model of a full twist



(b) Full twist on a geometric braid

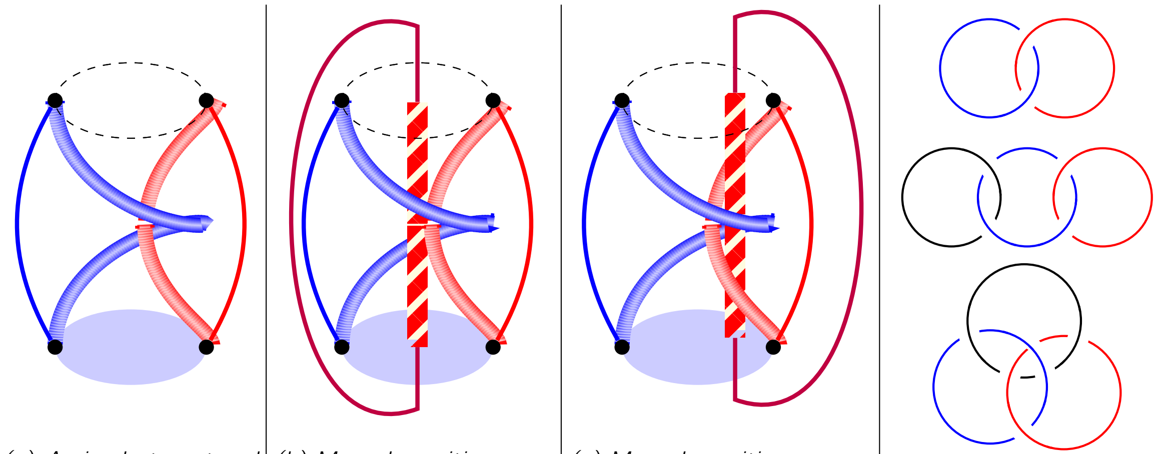


(c) Algebraic braid: full twist: RH diagram drawn as $\Delta.\Delta$

Fig. 3.7: The full twist

Let $\Delta^2\gamma$ be some geometric braid and consider its annular braid: if α is the annular braid corresponding to γ, then we have $\tau^n\alpha$ where n is the number of strands. By **T1**, we know that $\tau^n\alpha\tau^{-n} = s_n(\alpha)$ where s_k is the function that shifts each σ_i^ϵ in α to σ_{i-k}^ϵ (working mod n) and leaves any τ^k unchanged. Hence $\tau^n\alpha = s_n(\alpha)\tau^n = s_0(\alpha)\tau^n = \alpha\tau^n$, and so $\tau^n \in Z(\mathcal{CB}_n)$; furthermore the same argument applies to τ^{nk} for any integer $k \neq 0$. Equally, it is clear that if we have τ^m where n does not divide n, we will not in general have $\alpha = s_m(\alpha)$.

Now, suppose that $Z(\mathcal{CB}_n)$ contains some element β such that $\beta\alpha = \alpha\beta$ for all $\alpha \in \mathcal{CB}_n$, but $\beta \neq \tau^{nk}$. By **T1**, we have $\beta = \beta'\tau^k$ where β' has only σ_i generators, hence $\beta'\tau^k\alpha = \alpha\beta'\tau^k$ and so $\alpha = \tau^{-k}\beta'^{-1}\alpha\beta'\tau^k = s_{(-k)}(\beta'^{-1}\alpha\beta')$, i.e. $\beta'^{-1}\alpha\beta'$ is a rotation of α. But this implies



(a) A simple two-strand braid, and its closure

(b) Maypole position: centre

(c) Maypole position: off-centre

(d) Links for these closures

Fig. 3.8: Moving the Maypole does not result in conjugate braids

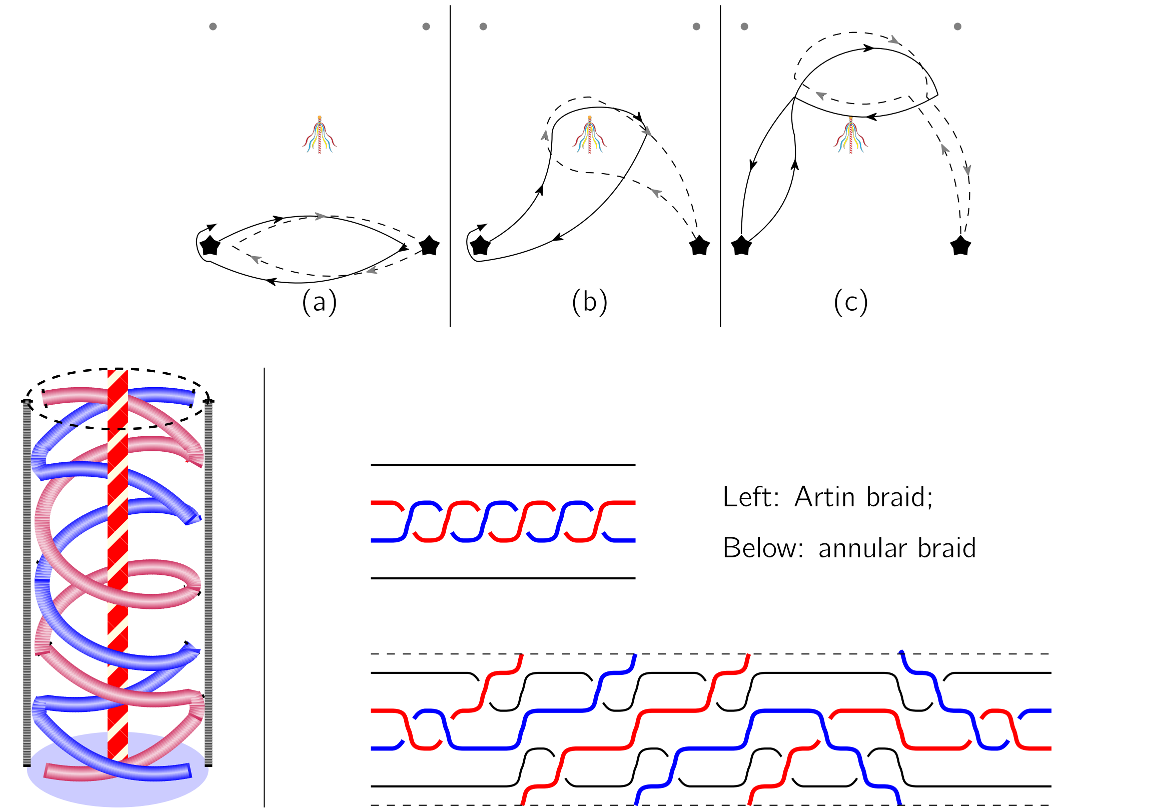
that $\beta' = \tau^k$, which is impossible since β' has no τ generators. Thus no such β can exist. $\square$

Finally, a note on conjugacy in the annular braid group more generally. It follows directly from the braid relations **A1** and **T1** that any single generator $\sigma_i \in \mathcal{CB}_n$ is conjugate to any other single generator σ_j. In the Artin braid group, we have the same result, but for the case $|j-i| = 1$ there is no single-generator conjugating element. Now consider $\gamma\tau = \sigma_i\gamma$: since none of the braid relations introduces or removes any $\tau^{\pm1}$, no γ can exist satisfying this equation, thus τ is not conjugate to any single generator. More general results concerning conjugacy in the braid groups have been the subject of much interest and although there are known solutions to the "conjugacy problem" for the braid group, once thought to be of potential interest for cryptographic schemes [Mah04], they are somewhat involved and beyond our scope here.

3.4 Positioning of the Maypole

In Sec. 2.3.1 we demonstrated the standard results that $\mathcal{GB}_n \simeq \mathcal{B}_n$, and that the Artin braids resulting from different projection angles for a particular geometric braid or dance figure are conjugate. Just as we can take different projection lines for a geometric braid, we could position the "Maypole" at different points on a surface (NB: this must be done *before* performing any isotopies). However, while the choice of a projection line for the Artin braid may seem arbitrary, in light of the shape of a dance set we do have a natural position for the Maypole, namely at the centre of the action, i.e. the unweighted barycentre of the strand "home positions".

Of course, a dance is not static, and the centre of activity often shifts from one end of the dance set to the other; we can see this as the Maypole moving relative to the action, and consider how the braid is affected. Firstly, moving the Maypole certainly does not generally yield conjugate braids. We can see this by looking at a slight variant of our two-strand braid,



Top: a sequence of trajectories for two dancers in a square set;

Bottom left: the trajectories as a geometric braid with Maypole; right: the Artin and annular braids

Fig. 3.9: Image showing a sequence of turns with equivalent Artin braids, but distinct Maypole braids

fig. 3.8. By moving the Maypole, we change the Maypole closure. Thus by Markov's Theorem, the corresponding two braids cannot be conjugate. Indeed, conjugacy is given by rotating the braid (i.e. cycling the strand numbers), just as in the Artin braids; this corresponds to changing the position of the "Maypole strand" in the isomorphism. □

What we can say is that the annular braid encodes information not just about the interaction between strands, but also about how each strand interacts with the (position of) the Maypole. Thus any crossing in the Artin braid may have been a crossing "in front of" or "behind" the Maypole, or a loop around the Maypole (fig. 3.9); not only this, but also a strand section with no other interactions may become a loop around the Maypole, as in the 1-strand braid in fig. 3.5. We will see examples of this in the circles and turns in Ch. 4, and we will be able to capture this extra information using Maypole-specific invariants in Ch. 5.

We can now formally justify placing the Maypole at the centre of the set: let p be the path traced by a strand (we consider a path as a series of t points sampled over the normalised time period; by defining a finite grid of possible points, we have a finite set of such paths); consider the Maypole as an observer of p; set $m = 0$ if the strand does not pass the Maypole, $m = 1$ if it does (we assume > 1 passes to be separate braid sections, or for $n = 1$ we

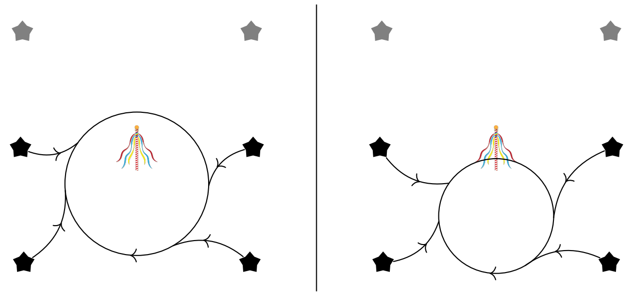


Fig. 3.10: Two dance figures we might consider equivalent to dance

count loops instead). We aim to position the Maypole such that if all paths are equally likely, the variable m gives us maximum information about the probable path p that was taken, i.e. the mutual information $\mathtt{I}(p, m) = H(m) - H(m|p)$ is maximised. Since we have defined the path distribution as fixed and uniform, this is done by maximising the entropy $\mathrm{H}(m)$, and it is a standard result in information theory that maximum entropy is achieved by the uniform distribution, $P(m = 0) = P(m = 1) = 0.5$, i.e. if the Maypole is passed by half of the (equally likely) paths – thus at the centre of the set. □

Arguably, this extra information is too much, since we know that the Artin braid group is isomorphic to the geometric braid group, thus the Artin braids already capture the full notion of isotopy. The two loops shown in fig. 3.10 might well be considered equivalent to dance. We might therefore wonder if we could keep the all-round view of placing our observer at the centre of the set, but leave the Maypole itself out of the model. We look briefly at this next.

3.5 Removing the Maypole

Suppose that instead of installing a Maypole on the dance floor, we imagine placing a swivel hook in the ceiling, with ribbons attached to the hook and to the dancers. The dancers can dance directly beneath the hook, with no fear of colliding with a Maypole. To model this as a group $\mathcal{H}_n$, we assume that each τ generator untwists itself, so that a half twist is indistinguishable from a full twist is indistinguishable from the identity.

Since, in the usual annular braid group, the relation **T1** enables us to write a braid β as $\beta \simeq \beta' \tau^k$ where β' has no τ generators and $k \in \mathbb{Z}$, we have that $\mathcal{CB}_n = \mathcal{H}_n \times \mathbb{Z}$; equivalently, $\mathcal{H}_n$ is the quotient of $\mathcal{CB}_n$ by the integers. These braids will differentiate between the three crossings of fig. 3.9, but not the two loops of 3.10, yet they lose information about the permutation, causing problems for composition of these braids (as in the example of a *half circle*). To preserve permutations, we might only allow our hook to spin back by full turns, i.e. quotient $\mathcal{CB}_n$ by the relation $\tau^n = 1$. Since we have $\tau^n = \Delta^2 = Z(\mathcal{CB}_n)$, this yields $\mathcal{H}'_n = \mathcal{CB}_n / Z(\mathcal{CB}_n)$. □

We will not consider this variant of the model further, but see e.g. [Haw12] for others who have used this model to think about braids as dances.

4 — Illustrations: braids and dances

We have already seen some examples in the introductory chapters. The goal of this section is to give a brief flavour of the typical braids and dances that we see in SCD. We introduce a few of the basic figures and then illustrate a selection of full dances.

4.1 Software

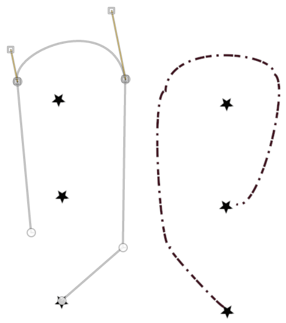
We were aware of various software which can work with Artin braids: `SageMath` [The22] is a powerful python-based mathematical software with rich group-theoretical functionality, including tools specific to the braid group and related Artin groups; `SnapPy` [Cul+24] is a useful tool for working with links and braid closures – for example, it is possible to draw a link by hand and send it to `SageMath`; `braidlab` [TB19] is a MATLAB package created for generating Artin braids from path data, with functions for equality, braid invariants, etc; it also has partial functionality for annular braids, but as it only supports the generator σ_n, not τ, and is not able to generate annular braids from path data, we found these functions to be of limited use.

4.1.1 Our pipeline

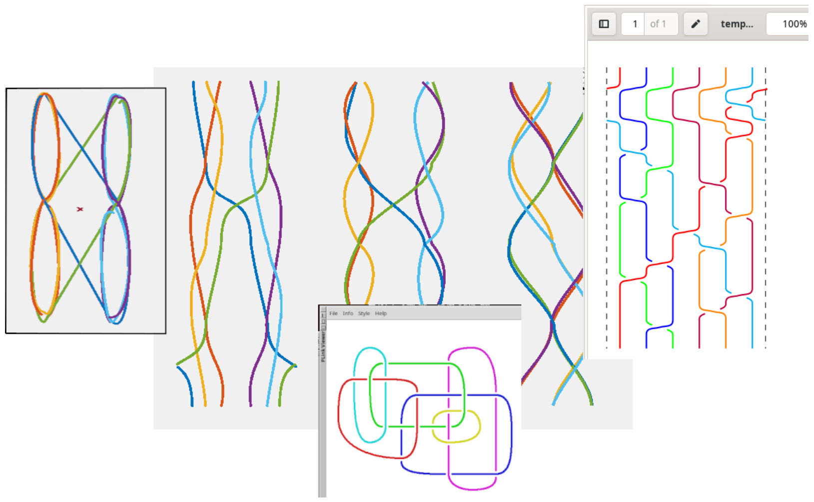
To support our investigation of annular braids for modelling SCD, we wanted a pathway to efficiently generate the braid for a given dance or figure. We implemented small scripts to convert Bezier curve data from the `gimp` path tool into 3-D trajectory data which can be read by `braidlab`, and extended this with MATLAB code to generate annular braids for the paths. Since many SCD dances make use of standard figures, these only needed to be plotted once, and could then be reused (possibly after rotation or reflection) in the braids for other dances.

To aid visualisation, since even `braidlab` does not have an intuitive way to view annular braids, we extended the LaTeX `braids` package [Sta22] to work with annular braids and added MATLAB scripts to visualise these by popping up a temporary PDF file. For practical use, we added commands to output braids in the formats understood by `SageMath` (vectors), the LaTeX `braids` package (in a `TikZ pic`), and the body of this LaTeX document (σ-notation).

With these tools, we can efficiently plot the paths of a group of dancers in `gimp`, as in the example in fig. 4.1a, and then generate Artin and annular braids, obtain properties of these



(a) Plotting paths in `gimp`



(b) Script output: various projections of a set of paths; PDF with braid; closure in `SnaPpy`

Fig. 4.1: Software pipeline

and their closures using `braidlab` and `SageMath`, plot the corresponding link in `SnapPy`, and output a PDF popup with a LaTeX visualisation (fig. 4.1). We also wrote code implementing the isomorphism f and $g = f^{-1}$ (using Artin combing), i.e. to convert between annular n-strand and Artin $(n+1)$-strand braids, along with various auxiliary functions (to animate a set of paths, remove a strand from a braid, double up a strand to model parallel tracks, calculate invariants, etc.). Future work could investigate the use of motion tracking software trained on a suitable dataset [Sun+22] to generate path data directly from a video of a dance. The relevant machine learning models are interesting in themselves, but beyond the scope of this project.

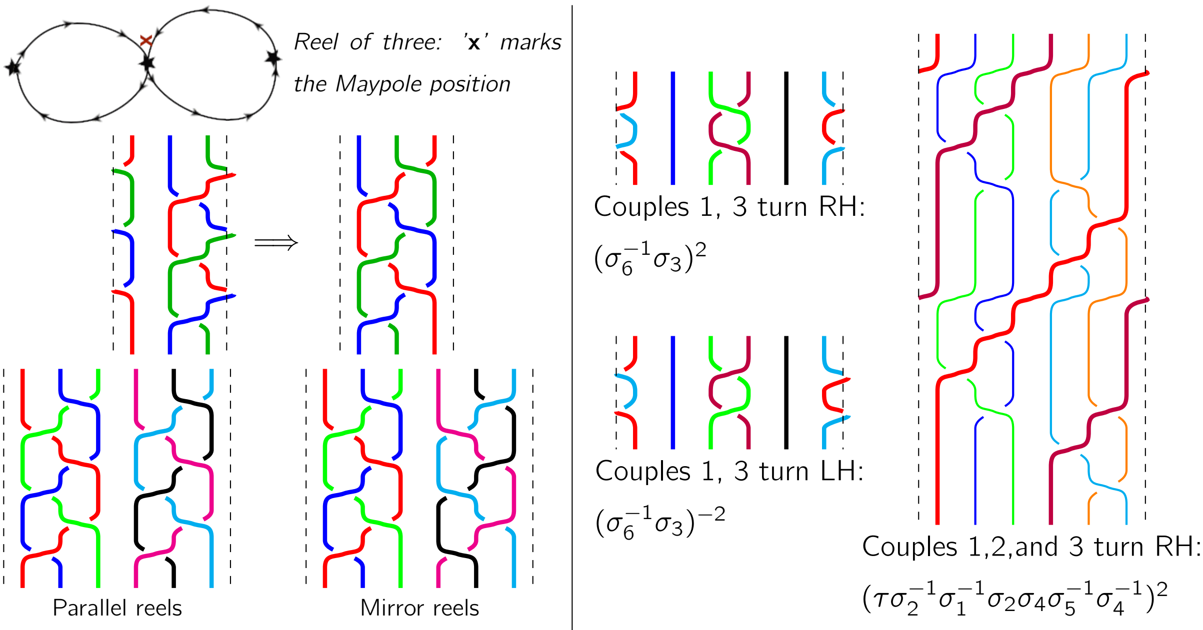
With the aid of this pipeline, this chapter illustrates the braids for a selection of dance figures and dances, to provide a flavour of the kinds of braids that are typical of SCD and highlight some interesting properties.

4.2 Dance figures

4.2.1 Circles – *right-hand star / left-hand star – chase*

We have already seen the *circle one way* in fig. 2.13a. This figure, in which dancers face into the circle and take hands, is a relatively rare figure in SCD (the *circle round and back*, which has identity braid, is much more common), but other figures that have the same track are more commonly danced – for example, the *chase* or *right-hand / left-hand star* in which dancers face in the direction of travel. We will see a way of differentiating these in Sec. 5.2.5.

As well as the Maypole braid, we saw the Artin braid for the circular track; that is, the braid obtained if the observer is outside the circle. SCD dances often have smaller circles not involving



(a) Reel of three and its braid

(b) Turn by the left, turn by the right, three couples turn

Fig. 4.2: Simple examples

all the dancers; depending on how energetic the dancers are, these circles may or may not go round the Maypole, as in fig. 3.10; if they do not, their subbraid will look like an Artin braid. In *Pelorus Jack* we will see a braid for the case where the dancers do loop around the Maypole. In *The Sausage Machine* we will see a figure where *right-hand / left-hand stars* are danced in parallel at the two ends of a rectangular dance set, and thus both "outside" the Maypole.

4.2.2 *Reel of three* – Extended reels – *The grand chain*

Reel of three The *reel of three* is danced by three dancers following the same figure-eight path, starting at the three points on the axis of the eight. When viewed from the side, this figure generates the "canonical" braid such as we might plait in someone's hair (fig. 4.2a), which is known to have various interesting properties. In a dance, the *reel of three* might be danced "on the side" – so that the Artin y projection would generate the natural braid – or "on the end" – so that the x projection would generate the natural braid, or even along a diagonal (in which case either projection would generate the natural braid).

Since the *reel of three* is typically danced along one edge of the dance set, the Maypole observer will be outside the figure and so the annular and Artin braids will generally be the same; the advantages of the Maypole braid are that if two such reels are danced in parallel, they are clearly distinct in the braid (fig. 4.2a), and that the braid is equally clear whether the reel is danced on the sides or on the ends. Many variants on the *reel of three* appear in SCD. Firstly, notice that the middle dancer starts from a four-way crossroads on the track – the direction in which

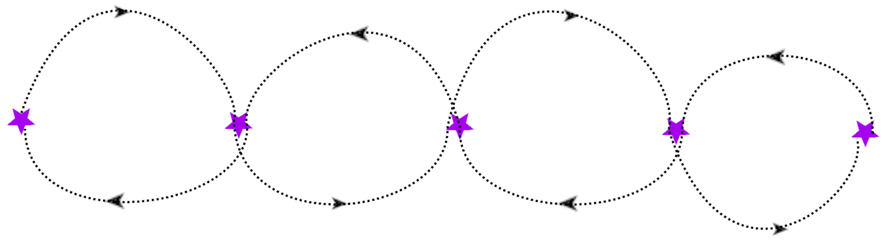


Fig. 4.3: The reel or "hey" of 5 dancers

they begin moving determines how the reel is danced. Thus there are four possible *reels of three*, with braids $(\sigma_1\sigma_2^{-1})^{\pm 3}, (\sigma_2^{-1}\sigma_1)^{\pm 3}$; these four braids are conjugate, but not equivalent: they form two pairs with translational and mirror image symmetry. Secondly, *reels* are often danced in parallel or with two sides mirroring each other; one couple or all three couples may progress through the reel as a unit; and so on. Fig. 4.2a shows *parallel reels* and *mirror reels*.

How might we achieve this "default" braid for $n > 3$ dancers? For the case n even, we have already seen the *grand chain* and its annular braid in fig. 2.13d. The *reel* can also be danced with an even number of dancers, and generates the canonical Artin braid as long as the observer, or Maypole, remains outside the field of action. For the case n odd, we can obtain a "canonical" Artin braid by extending the *reel of three* – a *reel of five* or more is rare in SCD, but sometimes seen as a *straight hey* in other country dancing styles (fig. 4.3). It is not possible to create a canonical annular braid with an odd number n of strands which is not also an Artin n-braid.

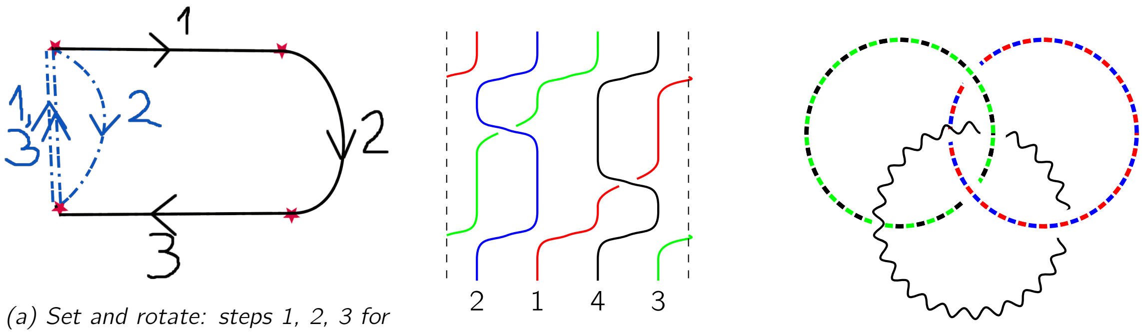
4.2.3 *Turn by the right / Turn by the left*

Two dancers extend their right (or left) hands towards each other and rotate once about the axis created by their joined hands, moving in the direction of their free hand (clockwise for the *right-hand turn*). These turns (fig. 4.2b) are very simple moves. Note, however, that if one of the turns happens to take place "around" the Maypole, we will see a different (sub)braid: this could happen in a dance if the lead couple has moved to central position, or if all three couples in a three-couple dance are turning in parallel.

4.2.4 A progression figure: *Set and rotate*

An interesting element of SCD is called "progression": at the end of the dance, couples are in a different home position from the one they started in. The dance can then be danced through again from the new positions. Fig. 3.1 represents one of the figures used to achieve progression, the *set and rotate* for four dancers (two couples) in a square. The dancers on one diagonal of the square will travel three-quarters of a square clockwise, while the dancers on the other diagonal travel up and down their own sides three times. At the end of the figure, the two couples have changed places (fig. 4.4a shows this path for the dancers starting on the left – the ladies' side). The annular braid for this figure is quite simple – we can see dancers 2 and

4 dancing up and down the sides without progressing "around the Maypole", while dancers 1 and 3 each follow three-quarters of a loop around the Maypole (fig. 4.4).



(a) *Set and rotate: steps 1, 2, 3 for two dancers*

(b) *Set and rotate annular braid*

(c) *Set and rotate – closure*

Fig. 4.4: Set and rotate

Here, although no single strand goes around the Maypole, the permutation means that the Maypole closure has two, not four, components from the strands, each of which is entangled with the Maypole (fig. 4.4c) – this tells us that, unlike the *reel of three*, there is nowhere we can cut open the cylinder of the annular braid to obtain an Artin braid. This closure has **chirality** – it is distinct from its mirror image, so the mirror image braid (in which the crossings are left-handed and the travel is anticlockwise) is not conjugate to the usual *set and rotate*.

4.2.5 Other progression figures

It is not uncommon for SCD dances to achieve the progression by the simple expedient of having dancers exchange places, as we will see in *Pelorus Jack* in the next section. Fig. 4.5 shows two other popular "progression figures": the *allemande* and the *poussette*. Unlike *set and rotate*, each couple moves as a unit in both these figures. As in *set and rotate*, no individual completes a loop around the Maypole, but it is nonetheless entangled in the Maypole closure.

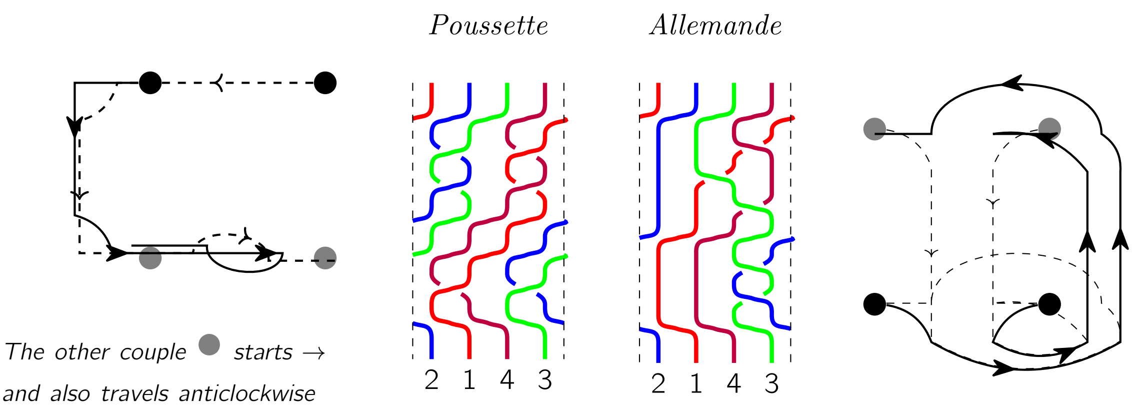
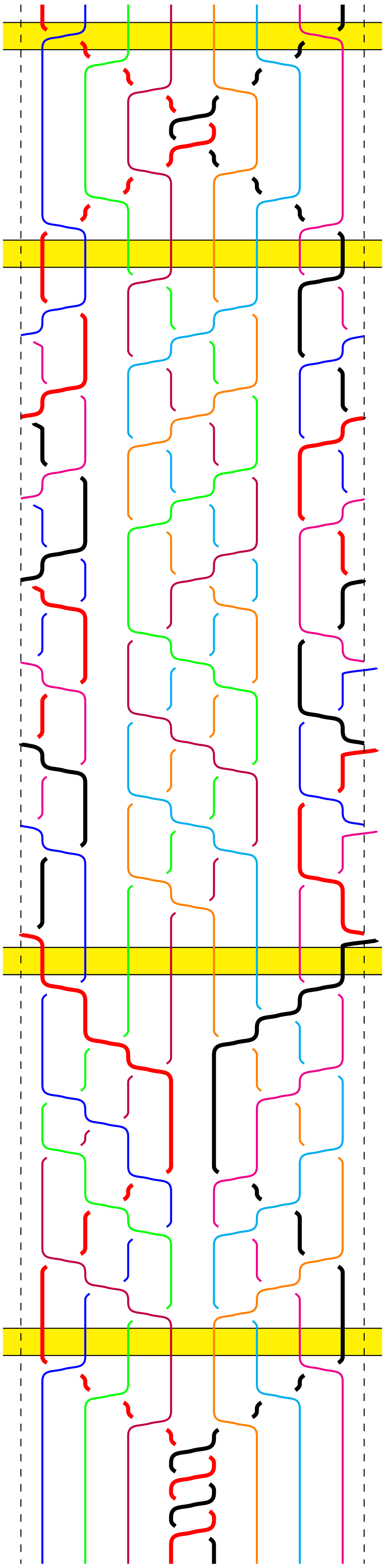


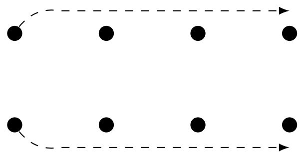
Fig. 4.5: Two progression figures: the poussette and the allemande

In the next section, we look at the full braids for three dances, beginning with the ceilidh dance *The Sausage Machine*.

4.3 Dances[1]

4.3.1 The Sausage Machine: a simple dance for four couples



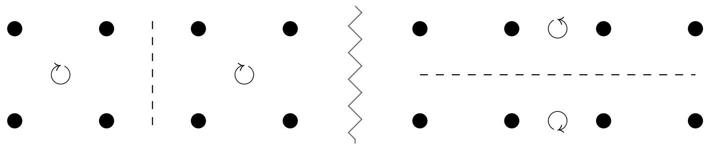


Phrase 1: Couple 1 dance down the outside of the longways set, turn, go back

Phrase 2: *Right-hand star* at each end;
Left-hand star at each end

This entire block is an identity braid!

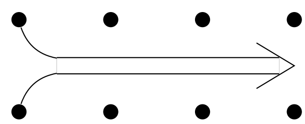
Most of the action in this braid is mirrored along the axis between the two lines of dancers. However, in this phrase, the split is between the two "ends" of the set, i.e. rotated by 90° (left figure below) – this causes the braid to loop around the Maypole, distinguishing it from an Artin braid



Phrase 3: The sausage! End couple form an arch and other couples dance through it and round back to place

This block is not *an identity braid: in fact, it is equivalent to two circles one way, one at each side of the set (right figure above)*

Phrase 4: Couple 1 dance down inside of set and spin together

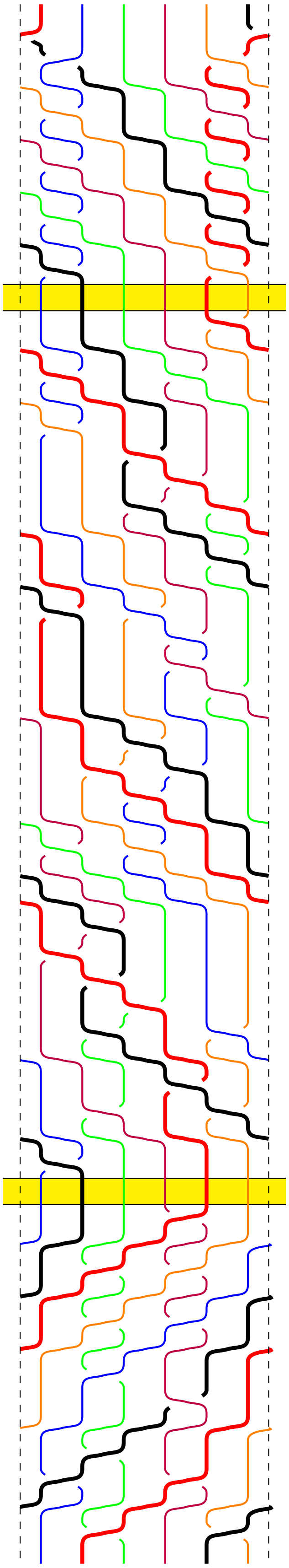


Each spin adds a twist between their strands, adding asymmetry to the dance!

Progression: couples now in order C2–C3–C4–C1: repeat x4

[1]Instructions and videos for the dances can be found in the `strathspey.org` database `https://my.strathspey.org/dd/search/dance` by searching the dance name.

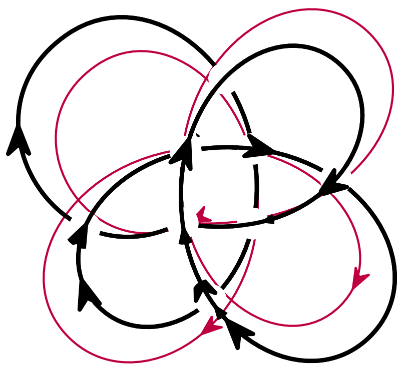
4.3.2 Pelorus Jack: a dolphin-themed dance for three couples



Phrase 1a: C1 swap places with each other, and then with their neighbour: this swap creates the progression

Phrase 1b: *Right-hand star* (one way only)!

Phrases 2&3: dolphins: C1 dance a four-leaf clover figure in tandem, switching lead on each leaf of the clover (see figure); the other dancers weave through the figure on the diagonals, alternating waiting and moving



Phrase 4a: *Left-hand star*: balances *right-hand star* from beginning

Phrase 4b: C1 swap places: balances swap at start

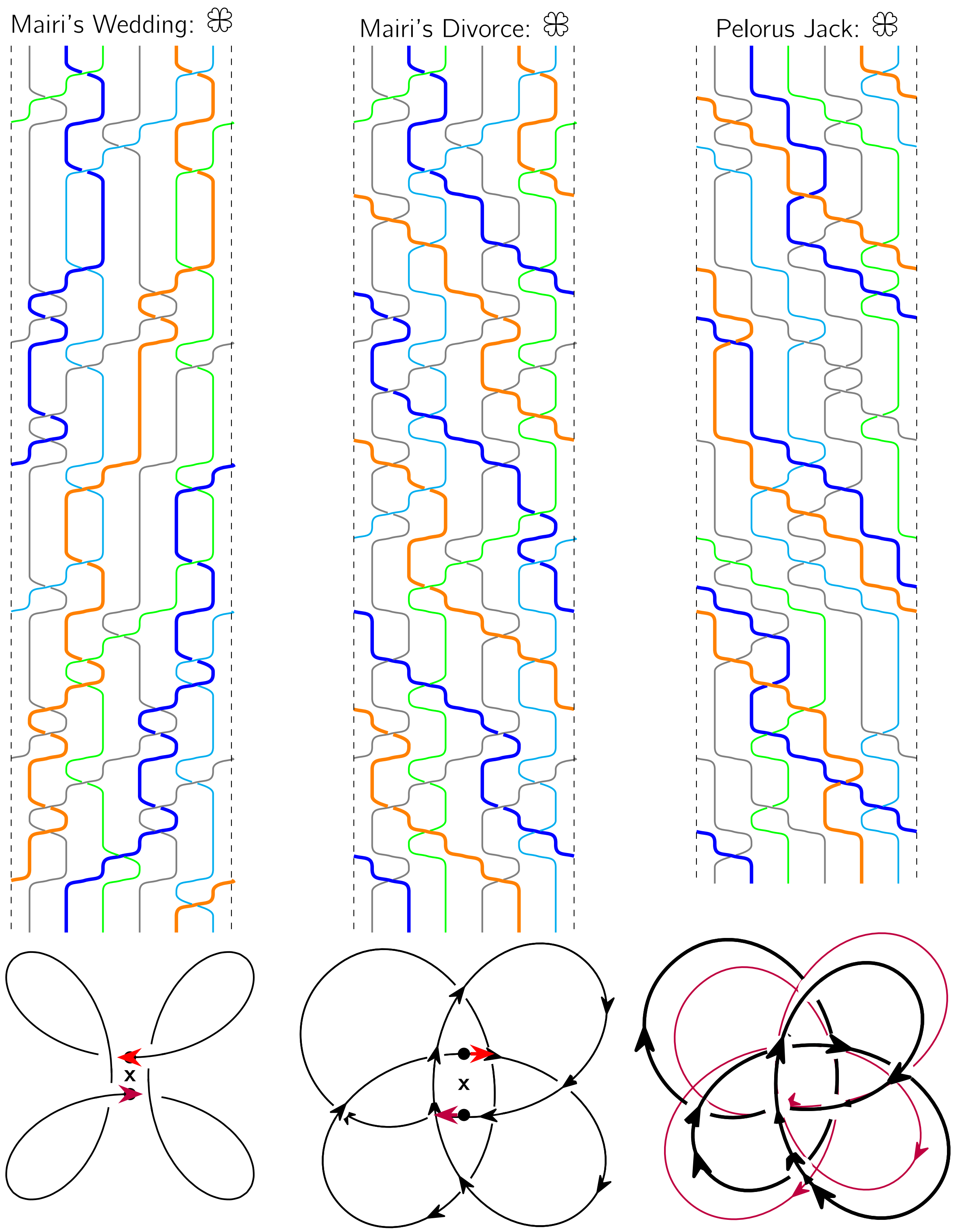
Progression: Couple ordering is now C2, C1, C3 (swap with neighbour in phrase 1 is not reversed here)

4.3.3 Close up: the "clover" figure

Half of *Pelorus Jack* is taken up with the "four-leaf clover" figure. In this dance, the figure is intended to represent a pair of dolphins playing – Pelorus Jack was a famous dolphin in Cook Strait, New Zealand. Looking at the track for the two dolphins, or dancers, as they trace out the figure, we can see that each of them makes a total of three loops around the Maypole: moving between each of the four corners of the clover, they make a half-loop around it, and the clover itself represents one complete loop. Were the dancers to cut the corners rather than complete the half-loops around the Maypole, then the figure would only involve one complete loop around the Maypole for each of the two dancers – and it would be anticlockwise!

Another popular dance, *Mairi's Wedding*, features just such a figure, but with the difference that rather than travel together like friendly dolphins, the newlyweds turn their backs on one another and embark on the clover on opposite diagonals. As they move between leaves, they turn left shoulders to each other. If this sounds rather unfriendly for a wedding dance – well, there is an alternative, commonly dubbed *Mairi's Divorce*! In this variant, the couple swing around each other by the right, and thus loop around the Maypole like the dolphins of *Pelorus Jack*. The next page shows a close up comparison of the three figures. In the braids, we can clearly see the additional loops around the Maypole, and the four "dolphins leaping" crossings of *Pelorus Jack*.

After that, we look at *The Summer Assembly*, a dance in a square set with no progression. Like the "clover" figures, this dance has a middle figure with net clockwise rotation: all eight dancers complete 1.5 rotations around the Maypole during bars 5&6. During the first four bars, the ladies travel halfway clockwise and the men anticlockwise. Therefore, the net rotation in the figure is 1 rotation for the men and 2 for the ladies (a total of 12 loops). This is the kind of information that we obtain from *braid invariants*, which will be the subject of the next chapter.

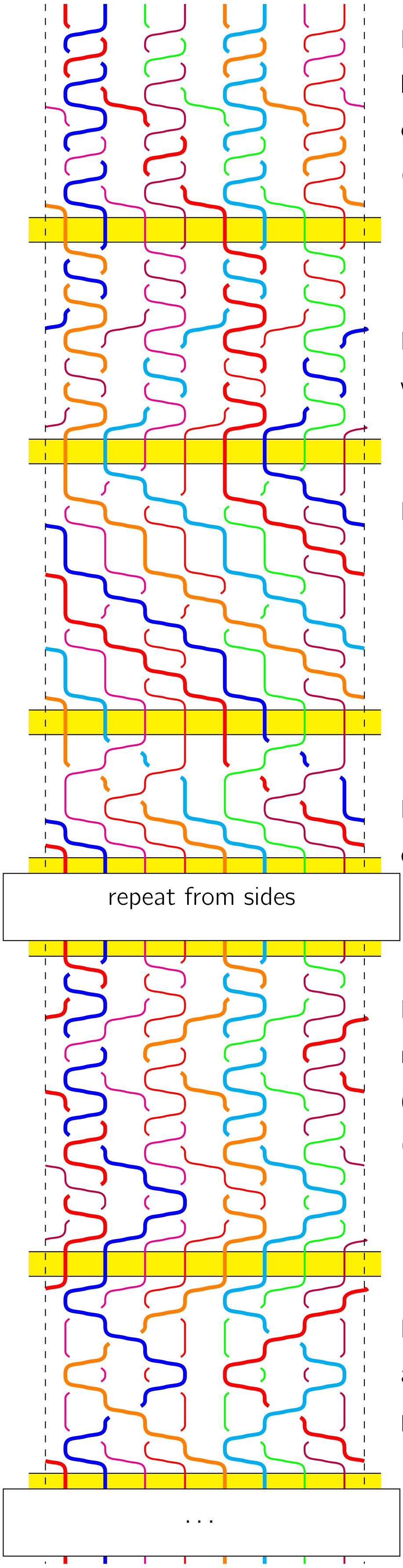


The *Mairi's Wedding* braid travels anticlockwise, while the other two travel clockwise.

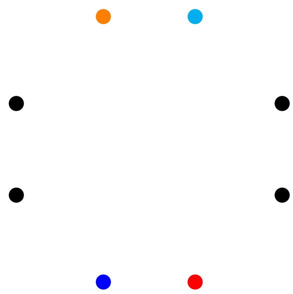
Look carefully at the track diagrams to see why.

Fig. 4.6: Comparison of the "clover" figures from three dances

4.3.4 The Summer Assembly: an 88-bar dance in a square set

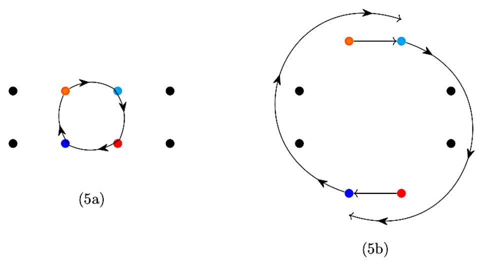


Phrase 1&2: *RH turn*, ladies move one place clockwise; repeat $((\sigma_1\sigma_3\sigma_5\sigma_7)^{-2}\tau)^2)$



Phrase 3&4: *LH turn*, men move one place anticlockwise; repeat. Everybody finds their partner!

Phrase 5a: End couples *right-hand star* (in middle)



Phrase 5b: Moving couples *chase halfway* in same direction, now outside the standing couples

Phrase 6: Other couples repeat

Phrase 7&8: Women loop round partner to right, then round neighbour to left; men repeat

(Sometimes called "tindering": swipe right, swipe left...)

$(\sigma_1\sigma_3\sigma_5\sigma_7)^2\tau^{-1}(\sigma_1\sigma_3\sigma_5\sigma_7)^{-2}\tau = (\sigma_1\sigma_3\sigma_5\sigma_7)^2(\sigma_2\sigma_4\sigma_6\sigma_8)^{-2}$

Phrase 9: End couples change places (making an arch) and back – we see one couple going inside the other couple in one direction, and outside in the other direction

Phrase 10 & 11: Side couples repeat; dance ends with a *circle round and back*

5 — Invariants

Diagrams like those of the preceding section are a nice way to visually abstract information from a set of trajectories, leaving only the elements that interest us. However, the true richness of the braid model comes from the group equivalence relations. A braid invariant is a function $\psi : \mathcal{B}_n \to X$ (or $\psi : \mathcal{CB}_n \to X$) for some set X, such that $\beta_1 \simeq \beta_2 \implies \psi(\beta_1) = \psi(\beta_2)$. This is a one-way implication; if the implication does hold in both directions, the invariant is said to be *faithful* (or *complete*). Thus $\psi\colon \psi(\beta) = 1$ for all $\beta \in \mathcal{B}_n$ is an invariant, albeit not a very interesting one, and not faithful. We have already seen the permutation function $\pi : \mathcal{B}_n \to S_n$: this is a more useful unfaithful invariant.

In this section, we look at ways invariants can tell us something about the braid – about its length or complexity, and about various "twistiness" properties – and consider how these might relate to dances.

Calculating invariants An algorithm for determining an invariant might be defined on the Artin braid word, or it might be based on a different representation of the braid, such as the geometric braid or a knot diagram. When we look at the invariant for an annular braid $\beta \in \mathcal{CB}_n$, our first question will be whether it makes sense to simply apply the algorithm to $f(\beta) \in \mathcal{B}_{n+1}$; if not, we briefly consider what we might do instead. Of course, for any Artin invariant ψ and any $\beta_1, \beta_2 \in \mathcal{CB}_n$, we have that if $\beta_1 \simeq \beta_2$, then $\psi(f(\beta_1)) = \psi(f(\beta_2))$; i.e. the invariant property holds. However, often we would expect of an annular braid invariant ψ that (say) $\psi(\sigma_n)$ behaves the same way as $\psi(\sigma_i)$, which may not be the case if we consider the isomorphic Artin braid.

5.1 Length invariants

5.1.1 Word length

Minimum Artin word length A simple first invariant: the **length** of a braid usually refers to the length of the shortest equivalent braid (in our usual σ_i generators) and has a natural equivalent for the annular braid group. Here, transformation to the "isomorphic Artin" can yield a different result. For example, the simple braids $\beta_1 = \sigma_1, \beta_2 = \tau$ have the same minimum word lengths, namely 1, but $f(\beta_1)$ has length 1 while $f(\beta_2)$ has length $n + 1$.

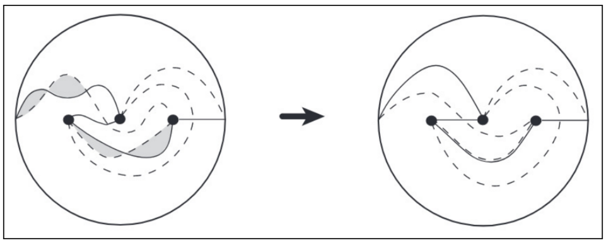


Fig. 5.1: Pulling tight superimposed curve diagrams (taken from [Deh+08, p. 195])

Alternative presentations The "isomorphic Artin" and the presentation $\mathcal{P}$ of the annular braid group can be considered as alternative presentations of $\mathcal{CB}_n$, giving us two different length invariants; we could further use the relation **T1** to define a new presentation of $\mathcal{CB}_n$ with just two generators s, t and hence obtain another invariant. There also exist quite different presentations of the braid groups and related groups. For example, in the **Coxeter presentation** of $\mathcal{CB}_n$, the relations are all symmetric[1]:

$$\begin{aligned} \mathcal{C} = \langle\ s_1, \ldots, s_n, t \mid\ & \textbf{A1, A2, Id} \\ & s_i t = t s_i && (i < n) \\ & s_n t s_n t = t s_n t s_n\ \rangle \end{aligned}$$

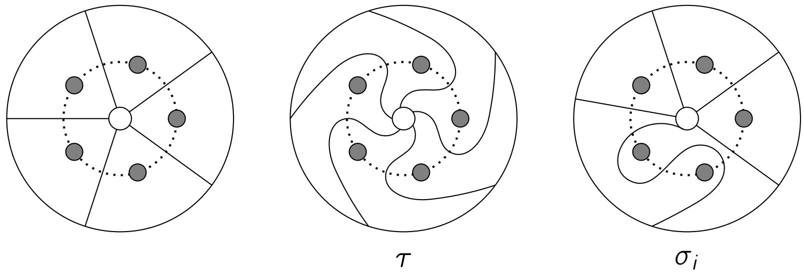
Normal forms Length of a minimal word is not the only way to measure the length of a braid. An Artin-combed braid is in a normal form, and its length can also be considered an invariant, since equivalent braids will have the same normal form. Other normal forms are based on alternative presentations of the group: the Garside presentation is derived for Artin braids from the half twist Δ and the simple permutation braids P_i we used for Artin combing, and leads to a normal form $\beta = \Delta^r P$ where P is a word in the P_i; this form was developed to solve the braid conjugacy problem ([Gar69]). For annular braids, the Garside normal form is derived from the Coxeter presentation, with fundamental word δ defined such that δ corresponds to a full twist. Mosher normal form [Mos95] uses triangulations of the disc to construct a canonical word for a braid seen as an action on the disc (Sec. 2.3.2).

5.1.2 Geometric complexity

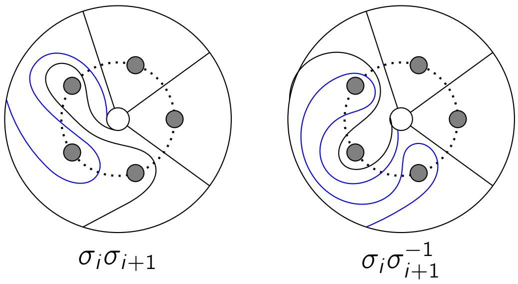
An alternative way of "measuring" a braid is from its curve diagram (Sec. 2.3.2). Given two curve diagrams, it is possible to superimpose them and perform an operation known in the trade as "pulling tight" (fig. 5.1 – braid topologists simply won't let bigons be bigons), such that if two braids are equivalent, their curve diagrams will pull tight into the same curve. By pulling tight a curve diagram against the identity, we obtain a canonical form of the curve, so that properties of the curve diagram are invariants of the braid. One such property is the **geometric**

[1] This is a presentation for the Coxeter group B_{n+1}, which has graph
[Cri99] gives a geometric proof that this group is isomorphic to $\mathcal{CB}_n$

Curve system of n arcs on the punctured annulus: our innovation is to replace the diameter with a closed curve as "border"



(a) Identity, action of τ and of single σ_i



(b) Action of $\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_{i+1}^{-1}$

$$\chi(1) = \log\frac{5}{5} = 0 = \chi(\tau)$$
$$\chi(\sigma_i) = \log\frac{7}{5} \approx 0.34$$
$$\chi(\sigma_i\sigma_{i+1}) = \log\frac{9}{5} \approx 0.59$$
$$\chi(\sigma_i\sigma_{i+1}^{-1}) = \log\frac{11}{5} \approx 0.79$$

(c) Complexities for the curve diagrams shown. As in the system of [DW07], $\chi(\beta)$ also depends on n.

Fig. 5.2: Our proposed system of **integrated laminations** *for a punctured annulus: the dotted line represents a "border", which plays the role of the diameter in the system of [DW07]*

complexity $\chi(\beta)$, defined for a canonical curve diagram based on a system of disjoint simple arcs, as in fig. 2.7f, to be the log of the number of times the curve c intersects the central diameter (the number of intersections grows exponentially with the braid [DW07]).

We recall that a curve diagram is a property of the algebraic, not the geometric, braid: different projections of the same geometric braid can have different curve diagrams. This means that the curve diagram for the Artin braid $f(\sigma_n)$ and thus its geometric complexity are different from those for, say, $f(\sigma_2)$. This result matches the implementation in `braidlab`, but does not reflect our intuition that the two crossings are equivalent. Just as there are alternative algebraic presentations, there are ways to construct and pull tight alternative systems of arcs and curve diagrams for an arbitrary compact surface [DW07], [Mos95], and thus for the punctured annulus; accordingly, we propose the system shown in fig. 5.2: in this system, $\chi(\tau^k) = 0$ for all k (a τ does not involve any interaction between strands), while $\chi(\sigma_i)$ is the same for all crossings σ_i (considered in isolation).

5.2 O what a tangled web we weave...

In general, dances that involve more weaving in and out will have higher complexity. Comparison of the dances on the list at a recent event[2] found those with more *reels* to have the highest complexity. However, in light of the braid relations, in particular **Id**, we must realise the length or complexity of a braid will not be directly related to properties such as the length in bars of a dance, the difficulty, or the energy expended, and it is not always easy to see how these

[2] `https://my.strathspey.org/dd/list/49594/`

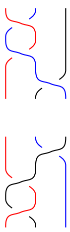
	$\lambda_{1,2} = \mathbf{-1}$	$\lambda_{1,3} = \mathbf{-2}$	$\lambda_{1,4} = \mathbf{0}$		$\lambda_{1,2} = \mathbf{-4}$	$\lambda_{1,3} = \mathbf{-1}$	$\lambda_{1,4} = \mathbf{-1}$
$\boxed{\Sigma(\lambda_{i,j}) = -8}$	$\lambda_{2,3} = \mathbf{-2}$	$\lambda_{2,4} = \mathbf{-2}$		$\boxed{\Sigma(\lambda_{i,j}) = -8}$	$\lambda_{2,3} = \mathbf{-1}$	$\lambda_{2,4} = \mathbf{-1}$
Projection angle: 0		$\lambda_{3,4} = \mathbf{-1}$		**Projection angle:** $\pi/2$		$\lambda_{3,4} = \mathbf{0}$

Fig. 5.3: Linking numbers for half circle - set and rotate – half circle back

invariants might be interpreted on the dance floor. We now move on to invariants more explicitly related to how the dancers interact with each other.

5.2.1 Linking number

The linking number $\lambda_{s_1,s_2}(\beta)$ of two strands s_1, s_2 of an Artin braid is a measure of how often they wind around each other. It is calculated from the braid diagram: we set λ_{s_1,s_2} to 0, and then walk down the braid. Each time s_1 and s_2 cross with some crossing $\sigma_i^{\pm 1}$, we add the exponent of the σ_i to λ_{s_1,s_2}. The braid β^{-1} has the same set of linking numbers (but negative) as the braid β, but unless β is a pure braid, these will not correspond to the same strands (e.g. $\sigma_1^2\sigma_2$: strands 1, 2 have $\lambda_{1,2} = 2$; for the inverse $\sigma_2^{-1}\sigma_1^{-2}$, it is strands 1, 3 that have $\lambda_{1,3} = -2$; see fig. below).



For the braid $\alpha = \gamma\beta\gamma^{-1}$, even if γ is a pure braid, the set $\Lambda_\alpha = \{\lambda_{i,j}\}$ will only equal the set Λ_β if β is also a pure braid. Thus two Artin projections (which we know to be conjugate) of a progressive figure may have different sets of linking numbers: as an example, fig. 5.3 shows the sets Λ_β for two projections of the sequence *half circle left – set and rotate – half circle right*.

We calculate linking numbers in just the same way for the annular braid, but the linking numbers we obtain are different: for example, the *circle one way* (fig. 2.13a) has the full twist Δ^2 as its Artin braid, in which all the exponents are positive and all linking numbers are 2, while the annular braid has no crossings and thus all linking numbers are 0.

A turn clockwise has linking number 2; anticlockwise -2. In phrases 1–4 of *The Summer Assembly*, dancers move to new partners for each turn, yielding a set of positive linking numbers and a set of negative linking numbers. By contrast, in phrases 7–8, every pair of dancers has linking number 0, since the men loop to the left, cancelling out the ladies' loops to the right. When a dancer loops around the Maypole "inside" outer dancers, they will typically pass each outer dancer once, adding ± 1 to that linking number; thus looking again at *The Summer Assembly*, we distinguish between the *right-hand-star* inside the standing dancers and the *chase* outside by negative and positive linking numbers.

Linking number is also defined for the closure of a braid, based on the crossings between pairs of components. Since taking the closure of an Artin braid does not introduce any new crossings, the linking numbers between components of a pure Artin braid ρ will be identical to the linking numbers between the corresponding strands of ρ. This applies only for pure braids: the two-strand braid σ_1 has linking number $\lambda_{1,2} = 1$, but its closure is the unknot. For an annular braid $\beta \in \mathcal{CB}_n$, we have seen that the Artin closure may add new crossings that are not in β and so, for example, the closure of the *circle one way* braid will have a set of non-zero linking numbers, corresponding to the difference between the annular and Artin braid linking numbers.

5.2.2 Writhe

Writhe is defined for an Artin braid as the sum of the linking numbers in the braid, i.e. the sum of exponents in the braid word; thus the writhe of the full twist is $n(n-1)$. Writhe is a geometric invariant of the braid and does not change on a choice of Artin projection. In fact, it has a physical intuition: if we make a pipe-cleaner braid held between two planes, as in fig. 2.3, fix one of the planes in place and free the other end, the braid will tend to try and untwist itself[3]. The writhe is a measure of how enthusiastically it will do so. This intuition can help us answer the question of how to calculate writhe for an annular braid β: the result should be the same as for a corresponding n-strand Artin braid (not the $(n+1)$-strand $f(\beta)$). Therefore, we need each τ to contribute $n-1$ to the writhe.

For example, in the two "bookend" phrases of *The Sausage Machine* only the *right-hand turns* contribute writhe, as they "twist up" these parts of the braids. The braid for the *sausage* phrase is two full twists of four in opposite directions, which cancel out . The other phrases contribute no writhe. Were we to look at the first half of phrase 2 in isolation, the two *right-hand stars* would have a writhe of $(3*4)+(3*4)$ – precisely the writhe of two full twists of four.

Similarly, in the first part of *The Summer Assembly*, as discussed, half the dancers accumulate positive linking numbers and half negative; again, these cancel each other out in the writhe. Phrase 5a has a full twist of four dancers: without the other dancers, Δ_4^2 would have a writhe of 12, and indeed this is the writhe of this section: each of the four inner dancers passes each outer dancer once, "undoing" these dancers' contributions to the four τ generators. In phrase 5b, we have a half twist of four dancers, now on the outside of the inner dancers, and $\frac{1}{2}\Delta_4^2 = 6$ is added to the writhe. The dancers in the repeat (phrase 6) also travel clockwise, so that the total writhe of these middle phrases is 36 – rather less than one full twist of eight dancers.

We look next at two new Maypole-specific invariants and their relation to existing invariants, and define a modification of the braid model to use linking number in a new way.

5.2.3 Winding number

The winding number $\xi(c)$ of a curve c about an axis is the net angle that a person standing at that axis would turn if they were to follow the curve from its start to its end; thus a curve that

[3]The author has verified this empirically!

loops once around the axis has $\xi(c) = 2\pi$; if c loops back the other way, $\xi(c)$ returns to zero, whereas if it continues around a second time, $\xi(c)$ will increase to 4π. While an Artin braid has no obvious axis, an annular braid does have one: we can take the winding number of a strand about the notional Maypole; this number will grow (or reduce) in steps of $\frac{2\pi}{n}$. To see that ξ is an invariant of an annular braid, we need only show that it is invariant under the equivalence **A2** (invariance is clear for **A1** and **T1**). This equivalence affects a subbraid of 3 strands $i, i+1, i+2$, thus with a width of 2 steps. The difference in position of each strand between the start and end of this subbraid is the step contribution to the winding number of that strand. Since **A2** is a braid equivalence and thus preserves permutation, the ending positions of the strands in this subbraid cannot be changed by applying **A2**, and so we have that ξ is an invariant. □

The winding number ξ_s of a single strand s corresponds to the (net) number of rotations that s makes about the Maypole. Thus in the full twist, every strand s has $\xi_s = 2\pi$. The lead couple in the "clover" figure of *Mairi's Wedding* couple each have $\xi = -2\pi$; in the *Mairi's Divorce* and *Pelorus Jack* figures, they each have $\xi = 6\pi$.

5.2.4 Net twist

Just as we summed linking number to obtain writhe, we can calculate the total winding number over all strands. Since we have fixed home positions assumed to be $\frac{2\pi}{n}$ apart, any generator σ_i will result in a net change of 0 to the total winding number, which thus depends only on the τ generators. We have already seen that we can always write a braid β as $\beta'\tau^k$ where β' has no τ^m; hence k is an invariant of the annular braid, which we can call **net twist** T_β.

Net twist is related to the writhe: when wound up and let go, this τ part of the braid would certainly spring back. However, it is a coarser measure, since a phrase like the *right-hand-stars* in *The Sausage Machine*, which we saw to have the writhe of two full twists, has $T_\beta = 0$. In *The Summer Assembly*, only phrases 5&6 contribute to T_β, and they each contribute $+6\tau$ – demonstrating this invariant does not distinguish between four dancers dancing a full circle, and eight dancers dancing a half circle; in combination with the braid permutation it would do.

5.2.5 Shoulder strands

We can use the linking number to create a new invariant *spin* for SCD: instead of one strand per dancer, we associate a strand with each shoulder of each dancer; these strands will cross when the dancer turns to face the opposite direction[4]. We add a constraint that pairs of strands must always move together. It is clear from fig. 5.4 that this constraint means it will always be possible to slide these "shoulder crossings" through the braid. This leads to a presentation of this new group $\mathcal{SB}_n$ with our usual τ, σ_i generators, plus a new set of generators $\omega_1, \ldots, \omega_n$: an

[4]Idea thanks to Hugh Griffiths, PhD (University of Edinburgh)

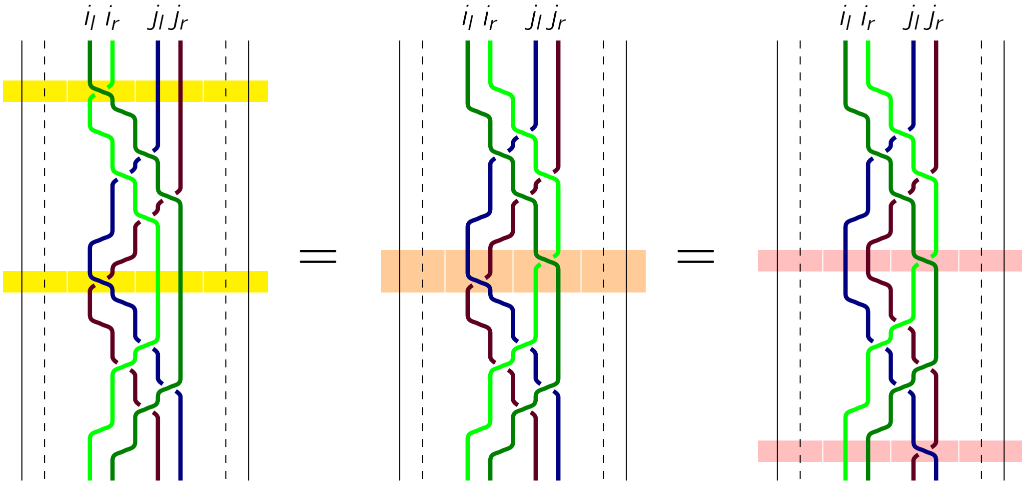
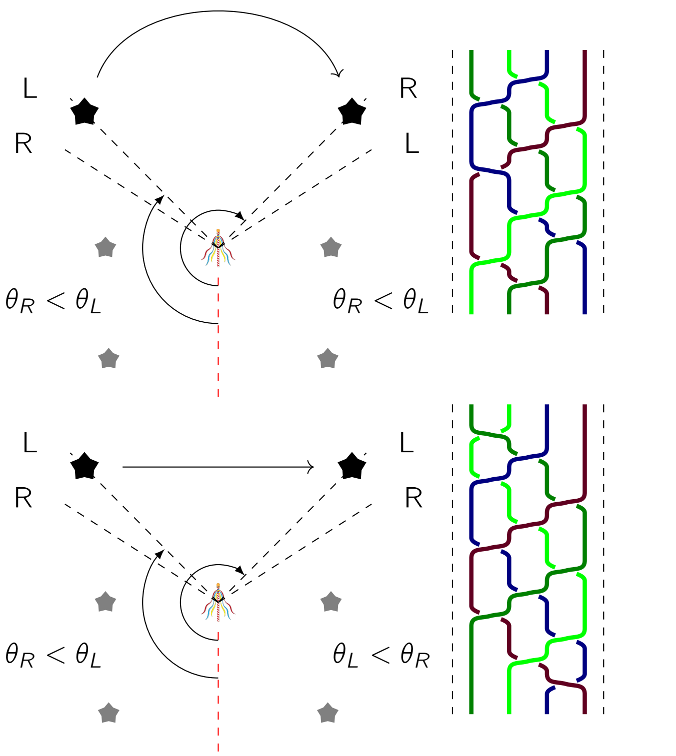


Fig. 5.4: Shoulder strands – equivalence relations

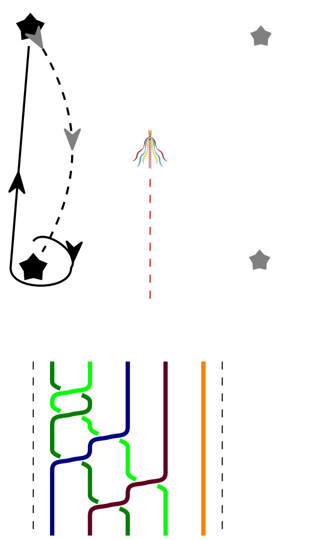
ω_i generator represents the braid action in which the shoulders of dancer i twist 180 degrees.

$$\mathcal{S} = \langle\ \tau, \sigma_1, \ldots, \sigma_n, \omega_1, \ldots, \omega_n \mid\ \mathbf{A1}, \mathbf{A2}, \mathbf{Id}, \mathbf{T1}$$
$$\omega_j\sigma_j = \sigma_j\omega_{j+1}; \omega_j\tau = \tau\omega_{j+1}; \omega_i\omega_j = \omega_j\omega_i\ \rangle$$

Like linking number, a spin is associated with a strand, not a position. In fact, it is none other than the linking number of the two "shoulder strands" associated with dancer i! However, since we can move ω_i terms through the braid, we can rewrite the braid in the form $\beta\omega_1^{k_1}\ldots\omega_n^{k_n}$, and then read off the net spin of each strand immediately. Thus we have that $\mathcal{SB}_n \simeq \mathcal{CB}_n \times \mathbb{Z}^n$. □



(a) Shoulder strands: right-hand turn (top); do-si-do (bottom)



(b) Shoulder strands detect looping outside the Maypole: cast up

Fig. 5.5: Shoulder strand invariant: examples

This model enables us to distinguish between a *do-si-do* and a *turn* (fig. 5.5a); between a *circle one way* (facing in) and a *full chase* (facing direction of travel). We can also distinguish a dancer looping outside the Maypole (fig. 5.5b). In the clover figure of *Mairi's Wedding*, the lead couple each build up spin on each quarter of the clover – loops made entirely outside the Maypole. Thus the strand spins can help us answer questions like: is there a general tendency towards turning in a clockwise or anticlockwise direction in a dance? Do some dancers spin more than others? Do the spins of the partners in a couple cancel out?

We now have several measures of the loopiness in a dance: one dancer turning on their heel has spin; one dancer looping around the Maypole has a winding number; two dancers turning around one another have a linking number; all these numbers contribute to the writhe and the net twist. These invariants can tell us about the balance of a dance – not only in terms of its clockwise bias or right-handedness, but also by allowing us to compare linking and winding numbers between dancers and couples.

5.3 Bar-hopping bewilders dancers

While we have succeeded in encapsulating some aspects of the dance in invariants such as net twist and linking number, other aspects continue to elude us. How energetic is the dance? Does the braid reflect the symmetries of the dance – reflective, translational, rotational? Essentially, when we start to examine the use of braids and braid invariants as an abstraction of an SCD dance, we realise that the braid relations do not well capture our notion of equivalence in a dance, in light of the freedom they give us to entirely restructure and rearrange a braid. Invariants are properties of the "whole dance", without taking into account its flow, yet dancing the same or similar moves in a different order generally feels like a different dance. The initial abstraction of simplifying a three-dimensional dataset of dancer trajectories into a short string of generators or diagrams like those in Sec. 4.3 is a useful way of compactly capturing information about the dance while throwing away irrelevant information. However, much of our intuition about the structure of a dance relies on its musical framework of bars, and the braid isotopies simply do not reflect this. If, acknowledging this problem, we remove the Artin relations (and **T1** from $\mathcal{CB}_n$), we are left with the free group on the generators – but even the identity relation is unsatisfying to a dancer enthusiastically springing along a *circle round and back*[5].

The next chapter outlines two different ideas for future study: firstly, we consider an alternative way to use braids and their closures to explore dances; secondly, we introduce groupoids as a generalisation of groups with potential for highlighting local properties and symmetries in SCD.

[5]Sometimes, a particularly energetic dancer will perform a full clockwise spin as the circle changes direction: using shoulder strands in the free group, this would be sufficient to distinguish the circle from the identity!

6 — New directions: closures and groupoids

6.1 Closures and devising dances

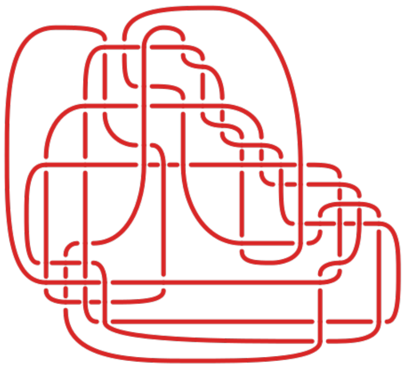
Calculating invariants of the braids for a collection of dances is one way of organising or comparing these dances. Another thing we can do is turn the question around, and ask which braids have some interesting property, and how might these braids be danced. For example, we might identify a set of dances whose braids are equivalent to the trivial braid, or whose braids all have the same closure. Recall that every knot or link can be represented as the closure of some Artin braid and there are known algorithms such as Vogel's algorithm [Gol17] for constructing such a braid. Here, we present a brief taster of this idea.

We will focus here on the Artin closure. Recall (Sec. 3.4) that we can transform any Artin braid β into some annular braid with the same Artin closure by constructing a geometric braid γ whose projection is β, choosing a central axis in γ, and projecting γ onto a cylinder. For a link L to be the Maypole closure of an annular braid, L must have at least two components, at least one of which is the unknot. The Artin braid β found by e.g. Vogel's algorithm will then have at least one strand that can be used as the "Maypole", and we can use the isomorphism g (after rotation if necessary) to find an annular braid $\alpha = g(\beta)$ with Maypole closure L.

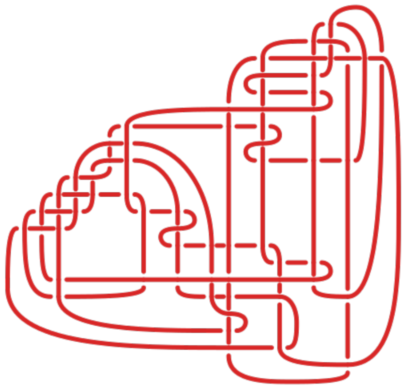
6.1.1 The trivial braid and trivial knot

In light of the fact that in SCD we often see figures teamed with their inverses – a *turn by the right* followed by a *turn by the left*, a *right-hand star* paired with a *left-hand star*, couples mirroring each other, etc. – we might expect to be able to find dances whose braid is equivalent to the trivial braid. However, we have mentioned that most SCD dances are progressive, which means that they do not generally have identity permutation and therefore cannot have a trivial braid. We might wonder whether k dance-throughs of the dance with braid β could become trivial, i.e. whether $\beta^k = 1$. If an element β of an infinite group has finite order, such that $\beta^m = 1$ for some $m > 0$, then β is called a **torsion element** of the group. However, the braid groups are known to be **torsion-free** [BB05]: they have no such elements, and so we are done.

Some dances, typically square-set dances like *The Summer Assembly*, are only danced once



(a) Knot closure of three-couple dance *Miss Agnes Lowden*: 1 component with 55 crossings



(b) Knot closure of four-couple dance *Sushi Rolls*: 1 component with 65 crossings

Fig. 6.1: Knots of *Miss Agnes Lowden* and *Sushi Rolls*: both include visible reel-like figures

through and do have identity permutation. We noted that *The Summer Assembly* has an overall clockwise rotation for all dancers – this tells us immediately that it cannot have a trivial braid. A tendency towards clockwise movement and an overall bias towards right-hand turns is something we saw in all three dances in Sec. 4.3 and is common in SCD. While we have certainly not investigated all 600+ non-progressive dances in the `Strathspey.org` database, we believe that this bias makes it unlikely that there will be a dance with the trivial braid.

We might instead consider taking the closure of the braid and asking if it is equivalent to the trivial knot (the unknot). This can be the case only if the permutation of the braid is an n-cycle, since otherwise the closure will have more than one component. Searching the `Strathspey.org` database by permutation ("progression"), we found just two dances with cyclic permutations: one six-person dance and one eight-person dance. We checked manually and neither *Sushi Rolls* nor *Miss Agnes Lowden* has the trivial knot as its closure (fig. 6.1).

6.1.2 Borromean Rings

We have observed that the *reel of three* yields the "canonical" three-strand Artin braid. This braid $(\sigma_1\sigma_2^{-1})^3$ is not only familiar to all of us from hairdos and rag rugs: it has interesting mathematical properties – among them, its closure (fig. 6.2a), a three-component link with the property that although the rings cannot be separated without cutting, removal of any one of the three leaves the other two components completely separate. This property arises because pairs of crossings between any two components have the same direction (both under / both over): the linking number of any two strands is 0. By building on the basic *reel of three*, we can create dances with closure variants related to the Borromean Rings:

Repeated *reel of three* If a team of three dancers dances the same *reel of three* R twice in a row, the link L_{R^2} that they will generate is a double Borromean Rings (fig. 6.2b): while the usual Borromean Rings have 6 crossings, L_{R^2} has 12. If they dance the reel k times, the resulting Borromean link will have $6k$ crossings. This occurs because each time the dancers follow the track around, they build up additional crossings with the same dancers as before. Of course, a dance consisting of consecutive *reels of three* would become rather tedious – but a dance that begins with a *reel of three*, followed by an identity-braid phrase such as *right-hand star, left-hand star*, followed by a second *reel of three* is conceivable.

Arbitrary numbers of dancers If two neighbouring dancers travel in parallel through the *reel of three*, then the closure of the corresponding braid will have four components. If, however, at the end of the *reel of three*, these two dancers swap places, the closure will have just one component for both dancers (fig. 6.2c). Since both dancers crossed over and under with the same partners, the link will still have the Borromean property. This corresponds to a fairly natural dance – not unlike the "dolphins" of *Pelorus Jack*. Three couples dancing in tandem is also a plausible SCD figure, and yields a related Borromean link. We can extend this idea to construct dances whose braid closure has the Borromean property with any number > 3 of dancers, by dividing the dancers into three teams. To close the link component of a team of more than two dancers, we use the obvious permutation braid: the dancers circle back to place.

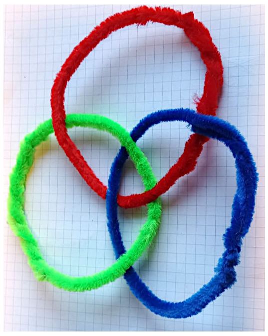
6.1.3 Brunnian links and braid index

The Borromean Rings can be generalised to Brunnian links, defined as any link of n loops (unknots) with the property that removal of any one of the components leaves a trivial link (i.e. the other components cease to be interconnected). These links have found some interest in chemistry, e.g. [LM94] [BS12]. Above, we explained the Brunnian property for the *reel of three* by the fact that all linking numbers are zero. The *reel of* n for n even does not have all zero linking numbers and thus certainly cannot have the Brunnian property. What about for $n \geq 3$ odd, for which the *reel of* n does have all zero linking numbers? This tells us that any group of three rings will be Brunnian, but not that the overall link (if $n > 3$) will be. For that, we need a concept of "higher order linking numbers"; these are known as **Milnor μ-invariants** [Mil54]. Specific links can be checked programmatically by software such as `SnapPy`.

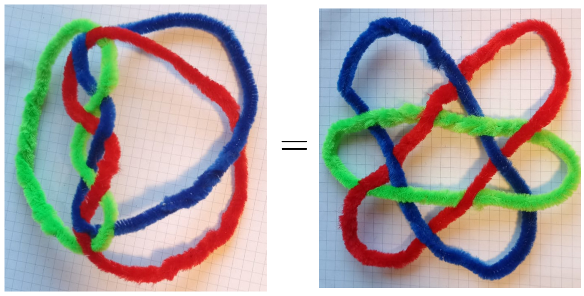
Two related families of Brunnian links are shown in fig. 6.2, the "horseshoe" (6.2d) and the "Bonio" (6.2e), with respectively eight and six crossings per component[1],[2]. It should be clear that an n-component Brunnian link must need *at least* n dancers. We can ask if it is possible to dance, say, a braid whose closure is the n-Bonio chain with *exactly* n dancers. To answer this question, we need the concept of **braid index** of a link: the minimum number of strands that a braid must have for its closure to be a particular link. If a link L has braid index k and a k-braid β, we can always construct m-braids $m > k$ with closure L, for example by the simple

[1]Horseshoe chain diagram by David Eppstein – CC0, `https://commons.wikimedia.org/w/index.php?curid=101988462`; Brunnian Bonio chain drawn with `SnapPy`

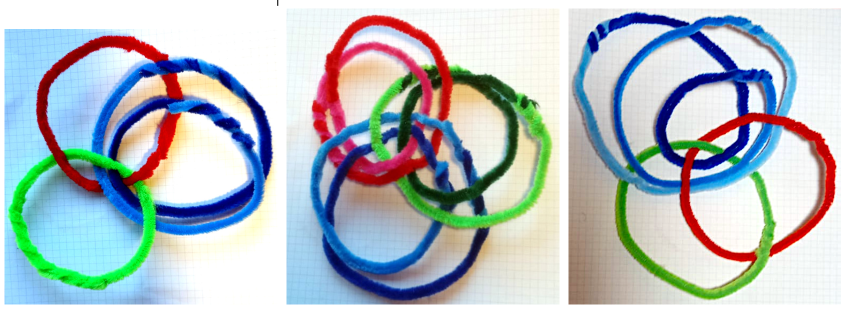
[2]A slight variant on the Horseshoe chain does not close the loop, but rather uses unknots at each end.



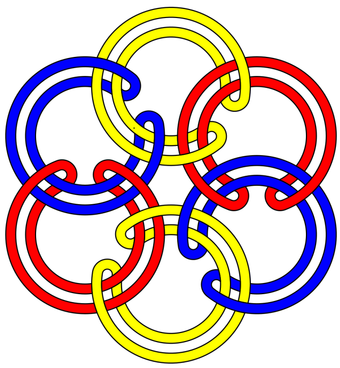
(a) The Borromean Rings



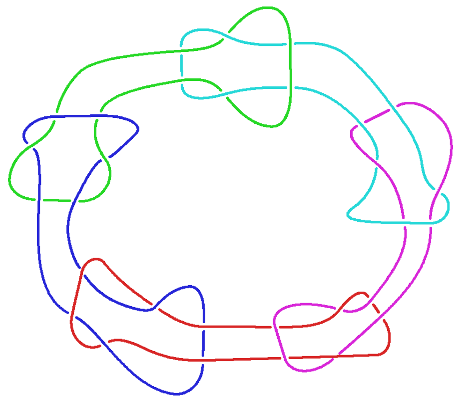
(b) Closure of 2 repeats of the *reel of three*, and an isotopic form of this link



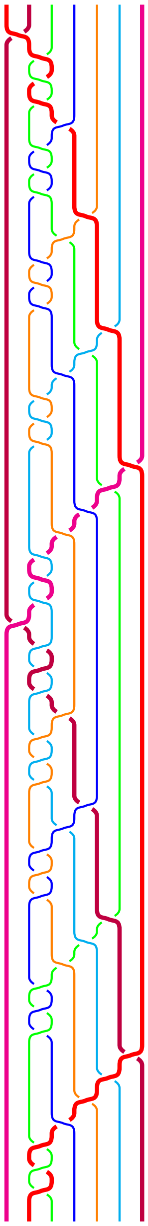
(c) Closure of *reel of 3* with one tandem couple; three tandem couples; one triplet



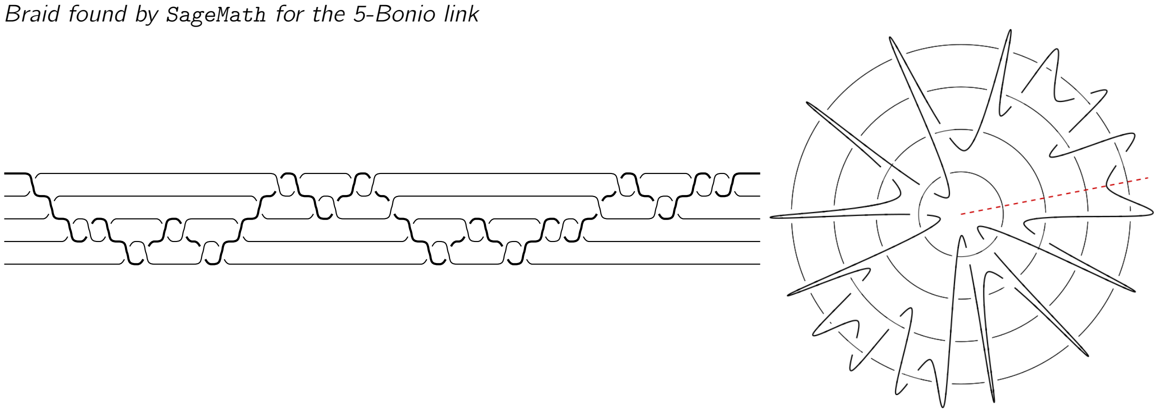
(d) A six-component "horseshoe" chain



(e) A five-component "Bonio" chain



Braid found by `SageMath` for the 5-Bonio link



(f) A simple Brunnian link and a corresponding braid (after application of **Id**)

Fig. 6.2: Borromean Rings and Brunnian links

expedient of inserting a strand $k+1$ and transforming $\beta \to \beta\sigma_k^{\pm 1}$. (This is called the **first Markov move** [MK99, Ch. 9]). However, there is no known way to determine the braid index of an arbitrary link. For the 5-Bonio chain, Vogel's algorithm (provided by `SageMath`) yields the 7-strand "Bonio braid" shown in fig. 6.2 (strands 1,2,7 form a cycle and are highlighted) – but does not guarantee to find the minimum number of strands. The braid index can, however, be bounded below using homology properties of the link [FW87], and it is possible to show that 7 is indeed the braid index of the 5-Bonio, and neither Brunnian chain (Bonio, horseshoe) of n components can be danced with n dancers.

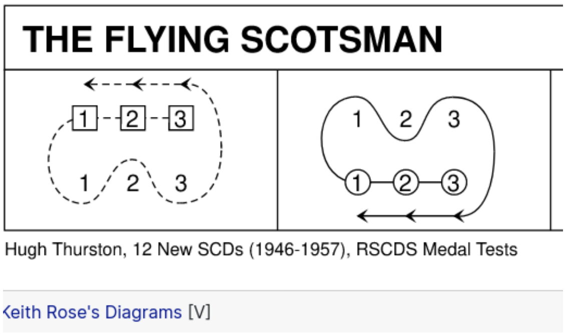
A third family of Brunnian links is shown in fig. 6.2f: a **simple Brunnian link** [GG07] does have an obvious n-strand braid β_{sb} (we cut open the circle, e.g. at the dashed line) – but it looks rather dull to dance! Indeed, it looks rather like the Artin-combed version of a braid: this is unsurprising since simple Brunnian links can be described by words corresponding directly to a presentation $\mathcal{A}$ of the pure braid group used in [Art47] to prove the validity of Artin combing.

Danceability Similarly, the "Bonio braid" above looks remarkably dull for the strand-7 (pink) dancer; furthermore seven is an unusual number of dancers. Due to the implementation of the isomorphism g using Artin combing, the annular braid $\in \mathcal{CB}_6$ whose Maypole closure is the 5-Bonio looks even less interesting! The "Bonio braid" features $(\sigma_2)^3$ repeatedly; energy could perhaps be added to the dance by adding an eighth dancer and inserting some parallel sections $\sigma_4^{\pm k}$ (in such a way that their inclusion is isotopic to the identity), and/or tugging at the strings ($\sigma_i^2 \to \sigma_i(\sigma_{i+1}\ldots\sigma_k)(\sigma_{i+1}\ldots\sigma_k)^{-1}\sigma_i$), equivalent to a dancer dancing up the set and back).

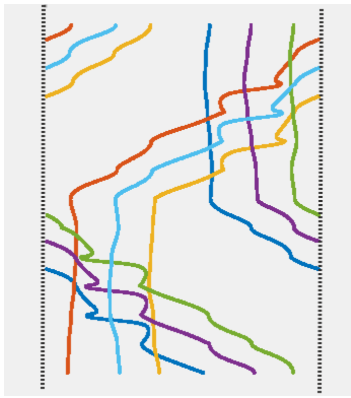
In general, exploring how to transform an algorithmically generated braid into an equivalent braid whose dance is more natural would make an interesting future direction – something akin to "reverse Artin combing". The braid invariant **step number** is the minimum number of rows needed for a braid diagram [Tia19, Sec. 7.2]. For example, the *grand chain* of n dancers has length $\frac{n^2}{2}$ and requires n rows. The length n braid $\sigma_1\sigma_2\ldots\sigma_n$ also has step number n. We speculate that, since parallelising crossings corresponds to more simultaneous activity, finding the braid $\beta_s \simeq \beta$ with minimum rows may be a starting point to devising an interesting dance.

A related invariant is the k-**danceability** of a link [ASS24]: the number of dancers who would be needed to dance along the path of a knot or link if the under-strand of each crossing must always be traversed first. Research into the relationship between k-danceability and braid index, and what this means for real dances, is ongoing. The authors of [ASS24] observe that some knots may require a dancer to wait for some time at an over-crossing – thus the same issue of finding a *good* dance arises, and there is certainly scope for investigation of the interplay between braid index, danceability, and a "good" dance.

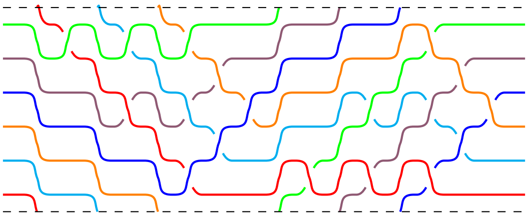
We look next at another model with potential for devising dances with certain properties.



(a) First half of The Flying Scotsman



(b) Annular projection of these tracks (plotted on "opened-up" cylinder)



(c) Braid of these tracks

Fig. 6.3: The Flying Scotsman

6.2 Groupoids

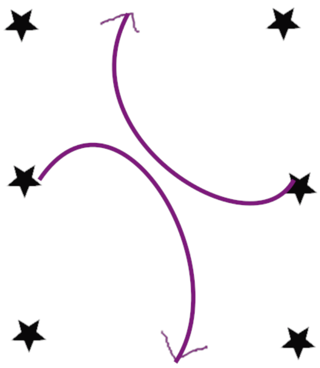
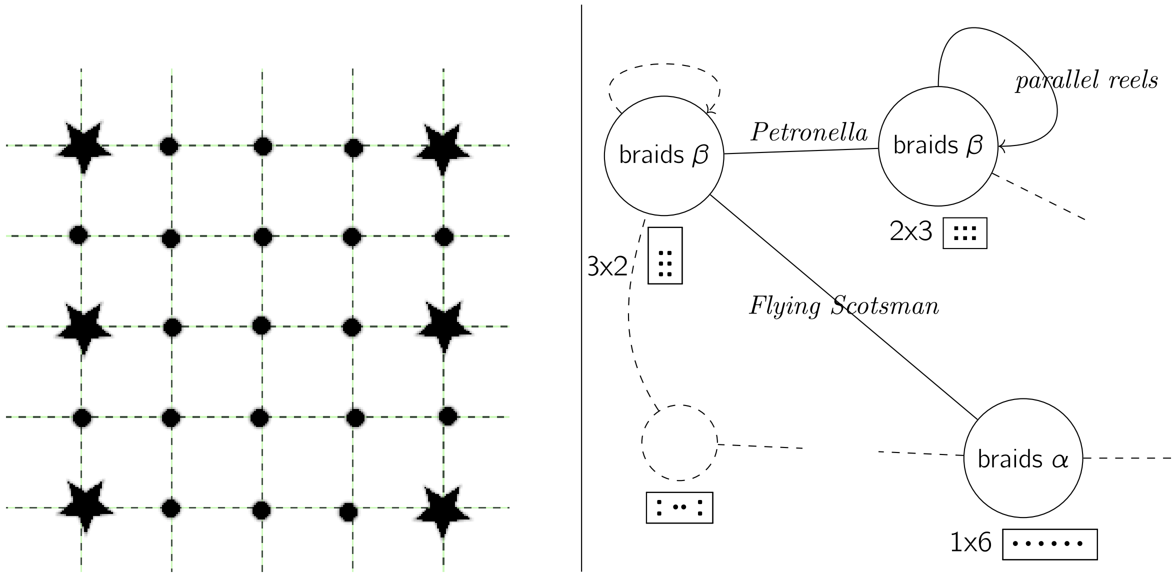


Fig. 6.4: The Petronella turn is one way to transform lines of 3×2 *to lines of* 2×3

Fig. 6.3 shows the dancers' crib diagram for the first two phrases of *The Flying Scotsman*, along with the annular projection of the trajectories and the corresponding braid. At the point when the dancers are halfway through the first phrase, and again halfway through the second phrase, all the dancers are crowded into one half of the dance set. Were we to pause the dance here, and insert *parallel reels of three* (fig. 4.2a), a crash would ensue – geometric braid condition 3 is not met at all. Dancer 3, for example (orange), appears in the braid to be moving throughout, although in the crib and projection, she is stationary for half the dance. *The Flying Scotsman* is no exception here. Many dances use moves such as the *Petronella turn* (fig. 6.4) that rotate the lines of dancers by 90° during a part of the dance. To some extent, these difficulties are already mitigated by the choice to use the annular braid, since we can easily follow the dance around the cylinder. However, an interesting alternative could be offered by the **groupoid** model. We sketch the idea below, although the details must be left for a future project.

Essentially, a groupoid is a way to capture situations where the configuration at the end of an action is different from the start. While in a braid, we assumed that each action (crossing) started and ended from the same configuration (n points spaced evenly around a circle), in a



(a) Possible dancer positions *(b) Visualising a groupoid*

Fig. 6.5: Example of a possible SCD groupoid

groupoid we can include actions such as the *Petronella turn*. As in a group, we require identity actions to exist, and we require that every action must have an inverse. However, composition is only possible if the respective configurations match.

6.2.1 Definitions

A groupoid comprises:

- A set of objects $\mathcal{C}$
- A set of transformations $\mathcal{T} = \{t : C_i \to C_j \mid C_i, C_j \in \mathcal{C}\}$ (we call C_i and C_j the "start" and "end" objects of t), such that the following hold:
  - **Identity**: For each $t : C_i \to C_j$, there exists $e : C_i \to C_j$ with $e.t = t$ and $t.e = t$
  - **Inverses**: Every $t : C_i \to C_j \in \mathcal{T}$ has an inverse $t^{-1} : C_j \to C_i \in \mathcal{T}$
  - **Associativity**: The product $t_i t_j$ is only defined if the end object of t_i is the same as the start object of t_j; if $t_i t_j$ and $t_j t_k$ are both defined then $(t_i t_j)t_k = t_i(t_j t_k)$

If $\mathcal{C}$ contains only one object G, then all $t \in \mathcal{T}$ have G as their start and end object, and we have a group. More generally, for any single object $C \in \mathcal{C}$, the set $\mathcal{T}_c = \{t : C \to C \mid t \in \mathcal{T}\}$ also generates a group. So the groupoid can be seen as a collection of groups and an explicit relation between groups.

6.2.2 The SCD "dancer positions" groupoid

In fig. 6.5a, the stars represent the original home positions of the dancers in the most typical SCD set. The dots represent other potential dancer positions at the end of a bar, or transformation. This gives rise to the following groupoid for the 3×2 dance set:

- An object $C \in \mathcal{C}$ is an unordered set of six distinct points on the grid.
- A transformation $t \in \mathcal{T}$ is one of the following:
  - A crossing $\sigma_{i,j} : C_s \to C_s$, where i, j are neighbouring points in C_s: we define "neighbouring" to mean that a straight line between i and j would not pass through any other point in this object[3];
  - A move $m_{i,j} : C_s \to C_e$, where $i \in C_s$, $j \notin C_s$, and there is a straight line from i to j that does not pass through any point in C_s. $C_e = \{j\} \cup (C_s \setminus \{i\})$;
  - A pass $p_{i,j} : C_s \to C_e$ or $q_{i,j} : C_s \to C_e$, where $i \in C_s$, $j \notin C_s$, and there is a straight line from i to j that passes through exactly one point in C_s. $C_e = \{j\} \cup (C_s \setminus \{i\})$; $p_{i,j}$ passes "behind" this point and $q_{i,j}$ "in front of" it.
- The inverse of any transformation $t_{i,j}$ is given by $t_{i,j}^{-1} = t_{j,i}$.

Just as we defined SCD figures as strings of crossings, we can now define them as sequences of transformations t. Any figure which corresponds to a true geometric braid is defined by a sequence of $\sigma_{i,j}$ transformations. The figure at the beginning of *The Flying Scotsman* is defined by a sequence of $m_{i,j}$ and $p_{i,j}, q_{i,j}$: there are no crossings. We can see a dance as a path between dancer configurations, with braid-like sections within each configuration.

6.2.3 The SCD "dancer directions" groupoid

In Sec. 5.2.5, we saw a way to model the spin of a solo dancer using pairs of strands. An alternative way to model spin would be to extend the usual annular braid into a groupoid by labelling each dancer's strand with the direction the dancer is facing. This kind of model is used in quantum computing, where particle interactions are represented by braids, and a spin is associated with each strand to create a groupoid [Kau94] [Goo23]. In this model:

- An object $S \in \mathcal{C}$ is a binary (or q-ary) string of length n, whose ith coordinate gives the direction that the strand in position i is facing (towards/away from the Maypole, or one of q compass directions);
- A transformation $\sigma : \{1, \ldots n\} \times C_i \to C_j$ or $\tau : \{1, \ldots n\} \times C_i \to C_j$ is equivalent to a braid crossing σ_i or twist τ but incorporates a specification of dancer directions; thus the crossing in a *do-si-do* is now distinct from the crossing in a *right-hand turn*.

Dancers can thus only perform the next action if they are facing the right way. One of the uses of this model could be to look for good flow in a dance: in general, sharp turns are considered awkward to dance. It would of course be possible to combine this groupoid with the previous groupoid, so that an object is $C \times S$.

[3]Note that this is a broader definition than the braid crossing, as it allows e.g. diagonal crossings

Continuing the theme of extending the braid model by assigning properties to "strands", we could endow the braid model with fixed strand properties: the couple number (so each number would be assigned to two "strands"), and/or the gender label. Models like this could enable us to "typecheck" that at the end of the dance, all the ladies are back on the ladies' side, albeit possibly in a different order, or that at the start of the figure known as a *ladies' chain*, the "strands" performing the chain are indeed ladies.

6.2.4 Applications of groupoids for SCD

What else might we do with such a groupoid? One thing we could do is bring the idea of "challenge dancing" and "challenge calling" from Square Dance to SCD! American Square Dance shares with SCD the property of having structured dance sets and a library of dance figures[4] – in square dance known as *calls* – which are chained together to create a dance that begins and ends in the home set formation. The definition of each call explicitly includes the configuration of dancers – including which direction they are facing – at the start of the call and the end of the call, thus each call triggers a transition between configurations and there is a natural mapping to a groupoid. In challenge dancing, the dance is treated explicitly as a mathematical puzzle to be solved on the fly [SH22]. We could also use the same techniques in a more relaxed setting for devising or checking dances.

A second way we could use groupoids is to explore symmetries. A point that stands out in SCD, when we look at the example dances in Sec. 4.3, attend a ball, or learn new dances in class, is that there is a lot of symmetry – in particular, different kinds of symmetry. *The Sausage Machine* has an axis of reflective symmetry along the set for most of the dance, but phrase 3 has translational symmetry across the set. In *A Summer Assembly*, we see translational and reflective symmetry in phrases 1–4, and rotational symmetry in phrases 5&6. The second phrase of *The Flying Scotsman* is a delayed mirror image of the first. Although we have found invariants to analyse some aspects of a dance, we have not really identified how our model of SCD can highlight these different forms of symmetry that appear in so many dances. Groupoids have local orbit structures analogous to orbits within groups, which are excellent for describing *local* symmetries [Wei96]. It is also possible to define equivalence relations on sequences of transformations [IR19, Ch. 3–4], and use these to quotient the groupoid; this technique has potential for exposing implicit symmetries.

While there was not scope in this project to explore this approach in any detail, we believe this to be a promising avenue for future investigations into the mathematics of SCD.

[4]See `https://www.tamtwirlers.org/taminations/` for an elegant overview

7 — Discussion

7.1 Summary

This project began with the goal of investigating braids as a model for Scottish Country Dancing, a dance style with regularities and symmetries that we believed would lend it well to mathematical study. Part of the charm of braid theory is its reach into numerous mathematical disciplines – topology, group theory, geometry, and beyond, and after introducing the theoretical background in Chapter 2, we used SCD as a source of examples to illuminate the less well studied annular braid from several of these angles, including introducing novel and adapted annular braid invariants (Chapters 3–5). In the course of the project we built on existing tools to create a software framework to support our explorations. By the end of Chapter 5 we had concluded that while the braid model can cast light on some elements of SCD, particularly in terms of overall balance in interactions and loops, the importance of the temporal flow of a dance makes braid isotopy inappropriate for many aspects. Finally, in Chapter 6 we dipped briefly into the idea of devising dances with certain properties, before introducing an extension to the braid model with promising possibilities.

7.2 Conclusions and future directions

The broad range of angles from which braids can be studied made this project something of a whistle-stop tour. We hope to have emphasised the annular braid as worthy of further study in its own right, particularly for applications involving trajectories of moving objects such as robot path planning. We alluded to AI-trained motion tracking models as a quite different, but also interesting, future direction for collecting such trajectories. In Sec. 6.1, we presented a taster of the interaction between braids and links, leaving open the tantalising question of how to make a braid "danceable", and a possible relationship to the knot invariant *k-danceability*. Finally, in Sec. 6.2, we outlined groupoids as algebraic objects that could extend the braid group and speculated that this model could present an interesting direction for further study into the maths of SCD, and in particular for exploring local symmetries. This latter direction seems to us highly promising for future SCD mathematicians. We look forward to seeing more SCD-related mathematical investigations in years to come.

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A — Artin relations

- **Artin 2**: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ $\quad\Longrightarrow$

Three-step relations:

$$\sigma_i^{\epsilon}\sigma_{i+1}^{\epsilon}\sigma_i^{\epsilon} = \sigma_{i+1}^{\epsilon}\sigma_i^{\epsilon}\sigma_{i+1}^{\epsilon} \qquad (\epsilon = \pm 1) \qquad \text{(same sign)} \tag{A.1}$$

$$\sigma_i^{\epsilon}\sigma_{i+1}^{\delta}\sigma_i^{-\epsilon} = \sigma_{i+1}^{-\epsilon}\sigma_i^{\delta}\sigma_{i+1}^{\epsilon} \qquad (\epsilon, \delta = \pm 1) \qquad \text{(one end differs)} \tag{A.2}$$

Two-step "chunk" relation:

$$\sigma_i^{\delta k}\sigma_j^{\epsilon} = \sigma_j^{\epsilon}\sigma_i^{\epsilon}\sigma_j^{\delta k}\sigma_i^{-\epsilon} \tag{A.3}$$

<u>Three-step relations:</u>

- A.1 follows directly from **Artin 2**
- A.2:

$$\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} = \sigma_{i+1}\sigma_i\sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_i^{-1} \qquad \text{(by A.1 twice)}$$

$$\boldsymbol{\sigma_i}^{-1}\boldsymbol{\sigma}_{\boldsymbol{i}+1}^{-1}\underbrace{\sigma_i\sigma_{i+1}\sigma_i}_{\text{A2}}\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} = \cancelto{1}{(\boldsymbol{\sigma_i}^{-1}\boldsymbol{\sigma}_{\boldsymbol{i}+1}^{-1}\sigma_{i+1}\sigma_i)}\sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_i^{-1}$$

$$\cancelto{1}{(\sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_{i+1}\sigma_i)}\cancelto{1}{(\sigma_{i+1}\sigma_{i+1}^{-1})}\sigma_i^{-1}\sigma_{i+1}^{-1}\boldsymbol{\sigma_i} = \sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}\cancelto{1}{(\sigma_i^{-1}\boldsymbol{\sigma_i})}$$

$$\sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_i = \sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1} \tag{A.2*}$$

A.2 follows by taking $(\text{A.2*})^{-1}$ □

<u>Two-step "chunk" relation:</u>

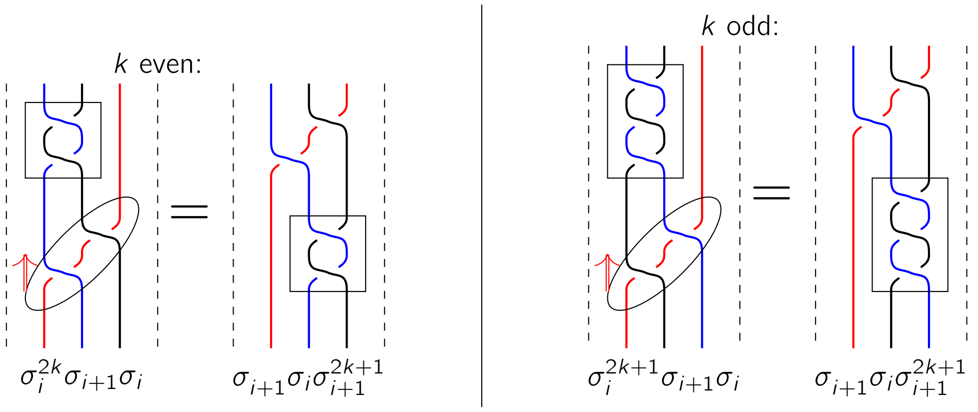
- $k = 1$: follows directly from three-step relations
- $k > 1$:

$$\sigma_i^{\delta k}\sigma_{i+1}^{\epsilon}\sigma_i^{\epsilon} = \sigma_{i+1}^{\epsilon}\sigma_i^{\epsilon}\sigma_{i+1}^{\delta k} \qquad \text{(visual proof below)}$$

$$\sigma_i^{\delta k}\sigma_{i+1}^{\epsilon}\cancelto{1}{(\sigma_i^{\epsilon}\sigma_i^{-\epsilon})} = \sigma_{i+1}^{\epsilon}\sigma_i^{\epsilon}\sigma_{i+1}^{\delta k}\sigma_i^{-\epsilon} \tag{A.3a}$$

$$\boldsymbol{\sigma}_{\boldsymbol{i}+1}^{-\boldsymbol{\epsilon}}\boldsymbol{\sigma_i}^{-\boldsymbol{\epsilon}}\sigma_i^{\delta k}\sigma_{i+1}^{\epsilon} = \cancelto{1}{(\boldsymbol{\sigma}_{\boldsymbol{i}+1}^{-\boldsymbol{\epsilon}}\boldsymbol{\sigma_i}^{-\boldsymbol{\epsilon}}\sigma_{i+1}^{\epsilon}\sigma_i^{\epsilon})}\sigma_{i+1}^{\delta k}\sigma_i^{-\epsilon} \tag{A.3b}$$

A.3 follows by combining A.3a and A.3b □



Larger "handle" relation:

$$\sigma_i^\epsilon\beta\sigma_i^\delta = \sigma_{i+1}^{-\epsilon}\dots\sigma_{n-1}^{-\epsilon}\beta'\sigma_{n-1}^{\epsilon}\dots\sigma_{i+1}^{\epsilon}\sigma_i^{\epsilon}\sigma_i^{\delta} \qquad (\delta,\epsilon=\pm1); \qquad \sigma_j^\epsilon\in\beta\to\sigma_{j-1}^\epsilon\in\beta'$$

Visual proof: