MATHS

VECTORS

Equation of a line:

$\underline{r} = \underline{a} + \lambda \underline{b}$

or $(\underline{r} - \underline{a}) \times \underline{b} = 0$

both directions!

or $\underline{r} = \underline{a} + \lambda(\underline{c} - \underline{a})$

$\underline{a} \times (\underline{b} \times \underline{c}) + \underline{b} \times (\underline{c} \times \underline{a}) + \underline{c} \times (\underline{a} \times \underline{b}) = 0$

$[a, b, c]$

Scalar triple product:

$\underline{a} \cdot (\underline{b} \times \underline{c})$ cyclic permutations

Lagrange's identity: $(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d}) \equiv (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c})$

Vector triple product

ppr to $\underline{a}$, in plane of $\underline{b}$ and $\underline{c}$.

$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$.

$(\underline{a} \times \underline{b}) \times \underline{c} = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{b} \cdot \underline{c})\underline{a}$

Distances:

from a point to a line $(\underline{a} + \lambda \underline{b})$

$|(\underline{p} - \underline{a}) \times \underline{b}|$

from a point to a plane $(\underline{r} - \underline{a}) \cdot \hat{\underline{n}} = 0$

$(\underline{a} - \underline{p}) \cdot \hat{\underline{n}}$ vector from pt to plane; cpt in normal dir.

between two lines $(\underline{a}, \underline{b})$ with arbitrary points $(\underline{p}, \underline{q})$ on the lines

$\left|(\underline{p} - \underline{q}) \cdot \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}\right|$ ← unit normal to both lines

from a line to a plane $(\underline{a} + \lambda \underline{b})$

unless ∥, 0

if $\underline{b} \cdot \underline{n} = 0$

$|(\underline{a} - \underline{r}) \cdot \hat{\underline{n}}|$ $\underline{r}$ is any point in the plane.

Equation of a plane:

$(\underline{r} - \underline{a}) \cdot \hat{\underline{n}} = 0$

or $\underline{r} \cdot \hat{\underline{n}} = \underline{a} \cdot \hat{\underline{n}}$

so $lx + my + nz = d$

or $\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a}) + \mu(\underline{c} - \underline{a})$

(plane contains $\underline{a}, \underline{b}, \underline{c}$).

so $\underline{r} = \alpha \underline{a} + \beta \underline{b} + \gamma \underline{c}$

with $\alpha + \beta + \gamma = 1$

Direction cosines

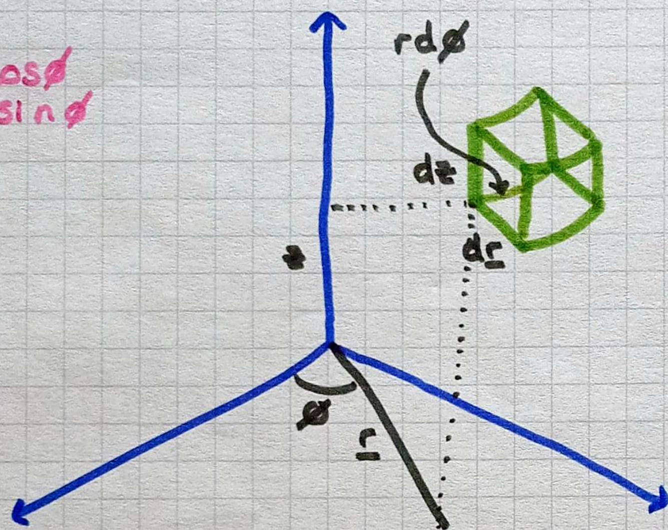
of $\underline{a}$ = $\frac{a_x}{a}, \frac{a_y}{a}, \frac{a_z}{a}$

Cosines of the angle made by a with each of the basis vectors

Polar Coordinates

Cylindrical

$x = r\cos\phi$
$y = r\sin\phi$
$z = z$



$\hat{\underline{e}}_r = \cos\phi\,\underline{i} + \sin\phi\,\underline{j}$

$\hat{\underline{e}}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j}$

$\hat{\underline{e}}_z = \underline{k}$

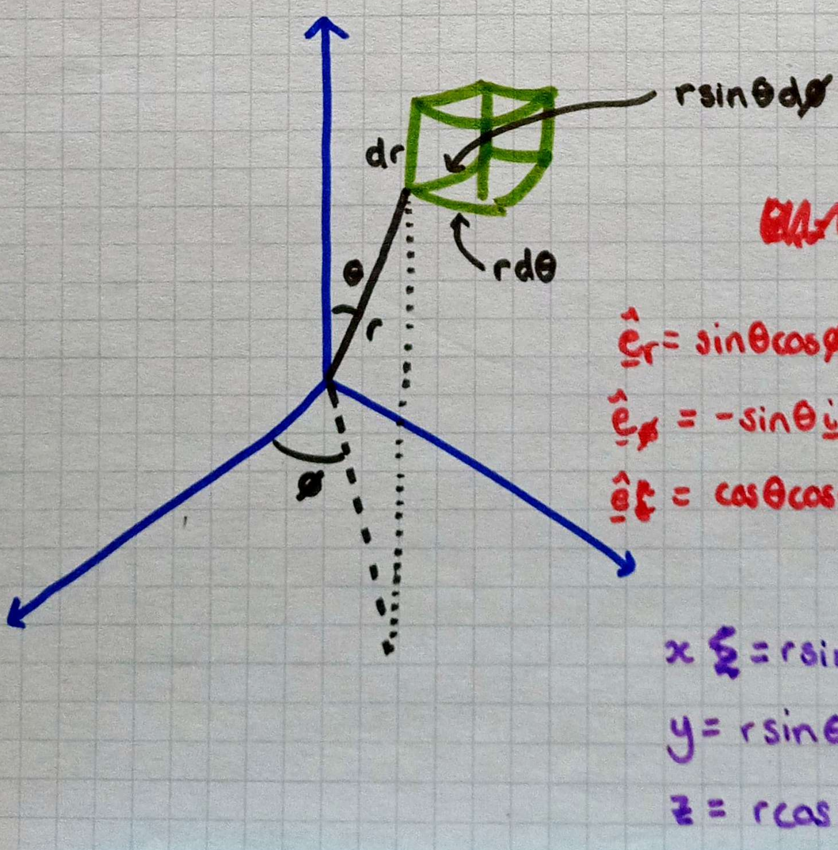
$\underline{i} = \cos\phi\,\hat{\underline{e}}_r - \sin\phi\,\hat{\underline{e}}_\phi$

$\underline{j} = \sin\phi\,\hat{\underline{e}}_r + \cos\phi\,\hat{\underline{e}}_\phi$

$\underline{k} = \hat{\underline{e}}_z$

$(\underline{r} = x\underline{i} + y\underline{j} + z\underline{k})$

Spherical



$\hat{\underline{e}}_r = \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k}$

$\hat{\underline{e}}_\phi = -\sin\theta\,\underline{i} + \cos\phi\,\underline{j}$

$\hat{\underline{e}}_\theta = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k}$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

Stirling's approximation:

$n! \sim n^n e^{-n}\sqrt{2\pi n}$

Schwarz's inequality:

inner product $\left(\int fg\,dx\right)^2 \leq \left(\int f^2 dx\right)\left(\int g^2 dx\right)$

or $(\underline{a}.\underline{b})^2 \leq a^2 b^2$

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

(Use $I^2 = \int_{-\infty}^{\infty} e^{-x^2}dx\, e^{-y^2}dy$; transform to plane polars) NB not rigorous!

Expand out $\int (f+\lambda g)^2 dx \geq 0$ look at determinant (quadratic in λ).

$\sim$ means "is asymptotic to"

CALCULUS

If there exists k such that $|f| \leq kg$ as $\lim(f)$ is approached, $f = O(g)$.

$$I(x) = \int_{t=u}^{t=v} f(x,t)\,dt$$
$$= g(x,v) - g(x-u)$$

$$\frac{dI(x)}{dx} = \int_u^v \frac{\partial f}{\partial x}dt$$: Leibniz' rule

A function is differentiable at x if the limit of
$$\frac{f(x+\Delta x) - f(x)}{\Delta x}$$
exists.

i.e. $\lim_{\delta x \to} \frac{O(\partial x)}{\delta x}$ = const or smaller

($O(\delta x)$ is ignored terms in expansion of $f(x+\delta x)$).

Leibniz' theorem
(derivatives of products)

$$f'' = uv'' + 2u'v' + u''v$$

and generalising

$$f^{(n)} = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} u^{(r)} v^{(n-r)}$$

proof by induction

Convergence of series

$\lim_{n\to\infty} u_n = 0$ (necessary, not sufficient)

$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) < 1$ (undefined if 0, divergent if >1)

For power series,
$|x| \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$ so dependent on $|x|$.

for two series P, Q:

sum, difference, product converges in region where both P and Q converge.

if both converge for all x, one can be substituted into the other to give a third convergent series.

Care with substitution otherwise!

POWER SERIES

Usually possible to differentiate and integrate.

N-R:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(derive from Taylor's thm).

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots a_nx^n$$

Limits: may vary according to direction of approach

$\lim f(x)g(x) = \lim f(x) \lim g(x)$

\+ \+

/ /

when dividing, if $f(x) = g(x) = 0$,
$\lim = \lim f'(x) / \lim g'(x)$.

Taylor's theorem:

$$f(x+h) = \sum_{n=0}^{\infty} f^{(n)}(x)\frac{h^n}{n!}$$

function must therefore be infinitely differentiable.

if $x=0$: Maclaurin series.

| Expansions of odd functions contain only odd powers of x! |

Complex numbers and Hyperbolic functions

inverses: $y = \sinh^{-1} x$ $(x = \sinh y)$

$\therefore\ e^y = \cosh y + \sinh y$
$= \sqrt{1+\sinh^2 y} + \sinh y$

.... $y = \ln(\sqrt{1+x^2} + x)$.

similarly for $\cosh^{-1} x$.

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

derivatives
(from definitions)

$\frac{d}{dx}\cosh x = \sinh x$; $\frac{d}{dx}\sinh x = \cosh x$.

$\cosh x = \cos ix$
$i \sinh x = \sin ix$

$\cos x = \cosh ix$
$i \sin x = \sinh ix$

hence Osborne's rule: trig identities remain unchanged but $\sin^2 x \to -\sinh^2 x$.

roots of a polynomial with real coefficients are real, or occur in conjugate pairs

De Moivre's theorem:

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
$= e^{in\theta}$

So $\quad \sin i\theta = \frac{1}{2} i(e^{\theta} - e^{-\theta})$

$\cos i\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$

$z^n + \frac{1}{z^n} = 2\cos n\theta$

$z^n - \frac{1}{z^n} = 2i\sin n\theta$

Use in solving equations in powers of cos/sin.

Logarithms: express z as $e^{i\theta}$
then $\ln z = i\theta$.

ORDINARY DIFFERENTIAL EQUATIONS

1. May be separable.

2. May be exact. (Use 2nd partial derivatives to determine).

Integrating factor: if $\frac{dy}{dx} + P(x)y = Q(x)$ ← linear

$\mu(x) = e^{\int P(x)\,dx}$

$\frac{\partial}{\partial y}(\mu A) = \frac{\partial}{\partial x}(\mu B)$

if $\mu = \mu(x)$ or $\mu(y)$, can multiply out.

3. May be homogeneous $\left(\frac{A(x,y)}{B(x,y)}\right.$; A and B have same degree $\left.= F(y/x)\right)$

sub in $y = vx$

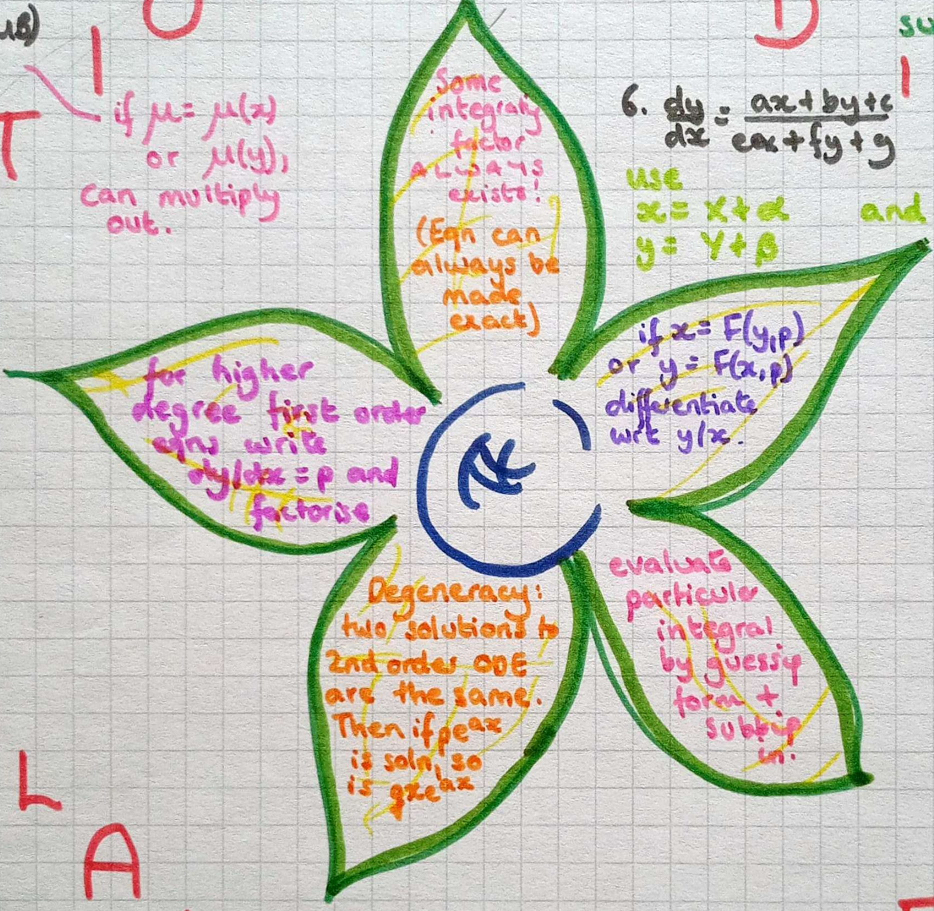
4. May be isobaric .... similar to homogeneous but substitution is $y = vx^m$ (same degree in initial eqn if y, dy have weight m)

5. May have form $F(ax+by+c) = \frac{dy}{dx}$

sub in $v = ax+by+c$

6. $\frac{dy}{dx} = \frac{ax+by+c}{ex+fy+g}$

use $x = X + \alpha$, $y = Y + \beta$ and $a\alpha + b\beta + c = 0$, $e\alpha + f\beta + g = 0$



eqn = 0 ⇒ particular integral ($e^{\lambda x}$)
eqn = f(x) ⇒ complementary function.
(in order f(x))

Must have as many unknowns as highest order.

if e^{ax} is solution to eqn=0 and e^{bx} is also solution, so is $e^{ax} + e^{bx}$ .... to obtain most general soln.
if $c(x)$ is soln to eqn = $f(x)$
$e^{ax} + e^{bx} + c(x)$ is also soln.

Partial Differentiation

etc.

Boltzman. Energy-level system with N particles, n_i in level i.

$$P = \frac{N!}{n_1!\,n_2!\ldots n_R!} \; ; \quad N - \sum_{i=1}^{R} n_i = 0$$

no. of particles

$$E - \sum_{i=1}^{R} n_i E_i = 0$$

energy

Maximise P by minimising denominator → can minimise $\ln\left(\frac{N!}{P}\right)$. Use Stirling's approx.

Stationary values:

$f_{yy} > 0$; $f_{xx} f_{yy} > f^2_{xy} \Rightarrow$ min

$f_{yy} < 0$; $f_{xx} f_{yy} > f^2_{xy} \Rightarrow$ max

$f_{xx} f_{yy} < f^2_{xy} \Rightarrow$ neither

(observe geom & physics first).

Lagrange multipliers

Condition : $g(x,y) = a \Rightarrow dg = 0$
$f(x,y) = \max \Rightarrow df = 0$

Multiply dg by λ and add to df. λ such that $\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x}\right) dx$ etc are independent. Solve.

Chain rule: for $f = f(x_1, x_2 \ldots)$
and $x_i = x(u_1, u_2 \ldots)$

(Changing variables)

then $$\frac{\partial f}{\partial u_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial u_j}$$

$$\left.\frac{\partial y}{\partial z}\right)_x \left.\frac{\partial z}{\partial x}\right)_y \left.\frac{\partial x}{\partial y}\right)_z = -1$$

Probability

CLT : for any large collection of independent random variables, the mean of the x_i ($\sum x_i / n$) has $E(z) = (\sum_i \mu_i)/n$
$V(z) = (\sum_i \sigma_i^2)/n^2$
tends towards Gaussian with μ, σ^2.

- Conditional: $P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(B)P(A|B)$

- $f(x) = P(X = x) = \begin{cases} p_i & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases}$: discrete

- $P(x < X \leq x + dx) = f(x)dx$: continuous

- Mean, $E(X)$, μ, $\langle x \rangle$: $\sum_i x_i f(x_i)$ or $\int_{-\infty}^{\infty} x f(x) dx$ — if the integral does not exist, no mean.

  $E(g(X)) = \sum_i g(x_i) f(x_i)$
  $E(aX) = aE(X)$
  $E(g(X) + h(X)) = E(g(X)) + E(h(X))$.

- Variance, $V(X)$, σ^2 : $E(X - \mu)^2$
  $= \sum_j (x_j - \mu)^2 f(x_j) \; / \; \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$

- Moments: $E(X^k) = \sum_j x_j^k f(x_j) / \int_{-\infty}^{\infty} x^k f(x) dx$
  $V(X) = E(X^2) - \mu^2$

- Binomial distribution (a number of independent trials with two possible outcomes).
  ${}^nC_x = \frac{n!}{x!(n-x)!}$ $\quad f(x) = {}^nC_x p^x q^{n-x}$

- Normal (Gaussian) distribution : reduce to standard form so
  $\phi(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$
  with $z = \frac{(x - \mu)}{\sigma}$ $\Rightarrow$ cumulative probability
  $P(X < x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}\left(\frac{u - \mu}{\sigma}\right)^2\right) du$
  (u is a dummy integration variable).
  - difficult to integrate. $\Phi(z)$ tabulated, then calculate.

- Approx to binomial: for large n, x, $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ ignoring lesser terms in expansion
  $f(x) \approx \frac{1}{\sqrt{2\pi n}} \frac{1}{\sqrt{p(1-p)}} \exp\left(-\frac{1}{2}\frac{(x - np)^2}{np(1-p)}\right)$: Gaussian $\mu = np$, $\sigma = \sqrt{np(1-p)}$
  continuity correction $x \pm 0.5$

Vector Calculus

$\frac{d}{du}(\underline{a}.\underline{b}) = \frac{d\underline{b}}{du}.\underline{a} + \frac{d\underline{a}}{du}.\underline{b}$

$\frac{d}{du}(\underline{a}\times\underline{b}) = -\frac{d\underline{b}}{du}\times\underline{a} + \frac{d\underline{a}}{du}\times\underline{b}$

Constant of integration is a VECTOR!

Scalar field: associates a scalar with each point in R; Vector field: associates a vector.

∇, "del," grad: $\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$

Gradient of a scalar field, ϕ $= \nabla\phi$

direction of fastest increase of ϕ.

Rate of change of ϕ in direction $\underline{r} = \nabla\phi.\hat{\underline{r}}$ $\left(\frac{d\phi}{ds}\right)$

Rate of change of vector field: use operator $\vec{a}.\nabla$
$= a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}$

(Vector fields)

Divergence: $\nabla.\underline{a}$
if $\underline{a} = \nabla\phi$ then
$\nabla.\underline{a} = \nabla^2\phi$: ∇^2 "Laplacian"

Curl: $\underline{\nabla}\times\underline{a}$

$\nabla\times\nabla\phi = 0$ (curl grad ϕ)
$\nabla.(\nabla\times\underline{a}) = 0$ (div curl $\underline{a}$).
i.e. if $\underline{a} = \nabla\phi$, curl $\underline{a} = 0$.

Polars:

(Cylindrical)

$d\underline{r} = \frac{\partial\underline{r}}{\partial\rho}d\rho + \frac{\partial\underline{r}}{\partial\phi}\partial\phi + \frac{\partial\underline{r}}{\partial z}\partial z$
$= \partial\rho\hat{\underline{e}}_\rho + \rho d\phi\hat{\underline{e}}_\phi + dz\hat{\underline{e}}_z$

(Spherical)

$d\underline{r} = dr\hat{\underline{e}}_r + rd\theta\hat{\underline{e}}_\theta + r\sin\theta d\phi\hat{\underline{e}}_\phi$

Line integrals:
$\int\phi dr$ or $\int\underline{a}.d\underline{r}$ or $\int\underline{a}\times d\underline{r}$
$d\underline{r} = dx\underline{i} + dy\underline{j} + dz\underline{k}$

$\oint_C Pdx + Qdy = \iint_R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$

(Green's thm in a plane)

note dV is scalar always!

Surface integrals
$d\underline{S} = \hat{n}dS$: $\hat{n}$ is unit normal, dS is surface elt.
(Sphere, radius a, $dS = a^2\sin\theta d\theta d\phi$).
can consider projection. Eg for projection onto x-y
$dS(\cos\alpha) = dA$
$\Rightarrow dS = \frac{dA}{|\hat{n}.\underline{k}|}$

flux: $\int_S\underline{a}.d\underline{S}$

$\Rightarrow$ if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$, sufficient and necessary
exact differential $\Leftrightarrow$ field is conservative
also if $\underline{a} = Pdx\underline{i} + Qdy\underline{j} = \nabla\phi$
i.e. $\nabla\times\underline{a} = 0$.

Vector area of a surface:
$\underline{S} = \int_S d\underline{S}$

Divergence thm:
$\int\nabla.\underline{a}dV = \oint_S\underline{a}.d\underline{S}$

also $\int_V\nabla\phi dV = \oint_S\phi d\underline{S}$
$\int_V\nabla\times\underline{b}dV = \oint_S d\underline{S}\times\underline{b}$

Stokes' theorem:
$\int_S(\underline{\nabla}\times\underline{a}).d\underline{S} = \oint_C\underline{a}.d\underline{r}$

also $\int_S d\underline{S}\times\nabla\phi = \oint_C\phi d\underline{r}$
$\int_S(d\underline{S}\times\underline{\nabla})\times\underline{b} = \oint_C d\underline{r}\times\underline{b}$

All functions can be written as sum of odd and even part $\Rightarrow f(x) = \frac{1}{2}(f(x)+f(-x)) + \frac{1}{2}(f(x)-f(-x))$

$= f_{even}(x) + f_{odd}(x)$

$\therefore$ Can represent any well-behaved function by a sum of sine and cosine terms.

If $f(x)$ and $g(x)$ are orthogonal

$$\int f(x)g(x)\,dx = 0.$$

$$\int_a^{a+L} \sin\left(\frac{2\pi r x}{L}\right)\cos\left(\frac{2\pi p x}{L}\right)dx = 0$$

$$\int_a^{a+L} \sin.\sin = \frac{L}{2} \text{ if } r=p \text{ else } 0$$

$$\int_a^{a+L} \cos.\cos = \frac{L}{2} \text{ if } r=p>0,\ L \text{ if } r=p=0 \text{ else } 0.$$

$$\int_a^{a+L} e^{\left(-\frac{2\pi i p x}{L}\right)} e^{\left(\frac{2\pi i r x}{L}\right)} = L \text{ if } r=p \text{ else } 0$$

FOURIER SERIES

Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty}\left(a_r\cos\left(\frac{2\pi r x}{L}\right) + b_r\sin\left(\frac{2\pi r x}{L}\right)\right)$$

$$a_r = \frac{2}{L}\int_p^{p+L} f(x)\cos\left(\frac{2\pi r x}{L}\right)dx$$

(p arbitrary) (often 0 or $-L/2$)

$$b_r = \frac{2}{L}\int_p^{p+L} f(x)\sin.\ dx$$

Dirichlet conditions:
- periodic function
- single valued and continuous (or finite number of finite discontinuities)
- finite number of maxima/minima in one period
- integral over one period of $|f(x)|$ must converge.

Derivation:

Since $f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty}(a_r\cos. + b_r\sin.)$

multiply by $\cos\left(\frac{2\pi p x}{L}\right)$

integrate over 1 period.

Parseval's thm:

$$\frac{1}{L}\int_p^{p+L}|f(x)|^2dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{r=1}^{\infty}(a_r^2 + b_r^2)$$

Complex series:

$$f(x) = \sum_{-\infty}^{\infty} c_r \exp\left(\frac{2\pi i r x}{L}\right)$$

$c_r = \frac{1}{2}(a_r - ib_r)$

$c_{-r} = \frac{1}{2}(a_r + ib_r)$

if $f(x)$ is real, $c_{-r} = c_r^*$

$$c_r = \frac{1}{L}\int_a^{a+L} f(x)\exp\left(-\frac{2\pi i r x}{L}\right)dx$$

(derive by multiplying by $\exp\left(-\frac{2\pi i r x}{L}\right)$ and integrating)