1 Topologies

Euclidean

$B(x,r) = \{y \in \mathbb{R}^n : ||x - y|| < r\}$

U is open if for every $x \in U$, there is some $\epsilon > 0$ with $B(x, \epsilon) \subset U$

Discrete

Open sets are all subsets of X

Indiscrete

(Trivial) Open sets are $\{\emptyset, X\}$

Particular point

based at x: set is open if it contains x

Cofinite

“finite complement”: open sets $U \subseteq X$ have $X \setminus U$ finite

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Cocountable

analogous

Gate

- X is functions from $[0, 1]$ to $\mathbb{R}$
- U is open if all the functions in it are “sufficiently close” at some finite set of gates.

Subspace

of $(X, \mathcal{T})$: given a space $Y \subseteq X$, $V = U \cap Y$ is open if U is open

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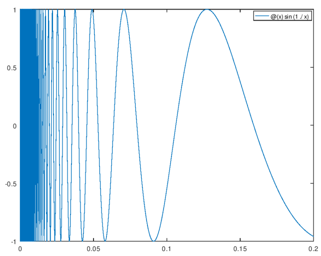
Topologist’s sine curve

$$Y = \{(0, y) | -1 \leq y \leq 1\} \qquad y = [-1, 1]$$
$$X = \{(x, \sin(1/x) | 0 < x \leq 1\} \qquad x = (0, 1]$$

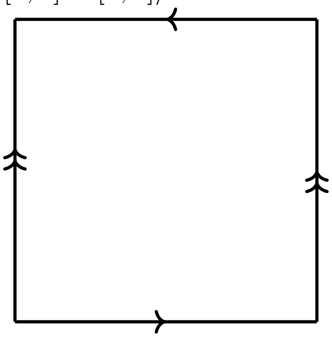


Quotient spaces

$q : X \to X/\sim$ is defined by $q(x) = [x]$; then

$$\mathcal{T}_q = \{U \subseteq X/\sim | q^{-1}(U) \in \mathcal{T}\}$$

$[0,1] \times [0,1]/\sim$:



for every $x, y \in [0,1]$:

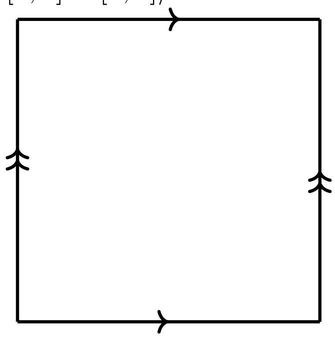
$(x,y) \sim (x,y)$ (reflexive);

$y \in [0,1]$:$(0,y) \sim (1,y)$.

$x \in [0,1]$:$(x,0) \sim (1-x,1)$

(a) Klein bottle

$[0,1] \times [0,1]/\sim$:



For $x, y \in [0,1]$:

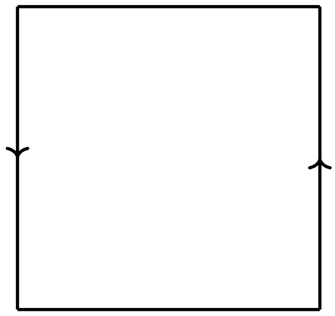
$(x,y) \sim (x,y)$. (reflexive)

$(0,y) \sim (1,y)$

$(x,0) \sim (x,1)$.

(b) Torus

$[0,1] \times [0,1]/\sim$:



$x, y \in [0,1]$:

$(x,y) \sim (x,y)$ (reflexive);

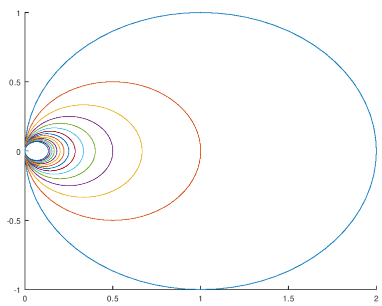
$y \in [0,1]$: $(0,y) \sim (1,y)$.

(c) Möbius

Polyhedron $|K|$ of a complex K has the subspace topology on $\mathbb{R}^n$

Hawaiian earring Circle of radius $1/n$ with centre $(1/n, 0)$:

$$R_n = \{(x,y) \in \mathbb{R}^2 | \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n^2}\}$$



2 Theorems and results

Ex 31. Compositions are continuous

Ex 36. Arbitrary intersections/finite unions of closed sets are closed

Ex 53. $(a, b) \times (c, d)$ are a basis for $\mathbb{R}^2$

Ex 57. Can provide infinitude of prime numbers using $S(a, b) = \{a+kb | k \in \mathbb{Z}\}$ as a basis of a topology

Ex 59. Homeomorphism is an equivalence relation

Ex 62. $\mathbb{R} \cong (0, 1)$ under usual topology (asymptotic function)

Ex 67. X Hausdorff $\implies Y \subseteq X$ Hausdorff

Ex 68. Discrete top: Hausdorff; indiscrete, pp not Hausdorff

Ex 92. Conn. cpts of $\mathbb{Q}$: sets of size 1

Ex 102. Union of finitely many compact sets is compact

Ex 128/129/130. Prod. top: $X \times Y$:

connected	$\iff$	X, Y connected	(+ Thm127)
compact	$\iff$	X, Y compact	(+ 124/125: Tychonoff)
Hausdorff	$\iff$	X, Y Hausdorff	

Ex 131. $[0, 1) \times [0, 1) \cong [0, 1] \times [0, 1)$: Start by projecting square out to disc

Ex 143. $X/\sim \;\not\Leftarrow\; X$ connected

Thm 4. Monotone convergence: increasing sequence bounded above converges

Thm 5. LUBs: non-empty set of real numbers bounded above has a least upper bound

Thm 6. Bolzano-Weierstrass every bounded sequence has a convergent subsequence [*Bolzano-Weierstrass property: infinite subset of a compact space must have a limit point*]

Thm 7. Intermediate value theorem (*becomes connectedness)*

Thm 8. continuous function on closed bounded interval is bounded and attains its bounds (*becomes compactness*)

Thm 108. X compact, $f : X \to \mathbb{R}$ cts: f is bounded and attains its bound

Lemma 46. A closed $\iff$ A contains all its limit points

Cor. 47. Closure $\overline{A} = A \cup \{\text{limit points}\}$

Thm 52. : basis if:

1. every $x \in X$ is in some $B \in \mathcal{B}$
2. for each pair $B_1, B_2 \in \mathcal{B}$ and each point $x \in B_1 \cap B_2$: there is some $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$

Lemma 71. $f : X \to Y$ X Hausdorff $\Leftarrow$ Y Hausdorff

Thm 81. Connected

$\iff$ only clopen subsets are the inevitable $\iff$ no surjective cts functions $X \to Y$ where Y is a discrete topology $|Y| > 1$ (*eg* $\{0, 1\}$)

Thm 82. Given an index set I s.t. for all $i \in I, A_i \subseteq X$ is connected, if there is a point in the intersection $\bigcap A_i$: $\bigcup A_i$ is connected

Thm 83. non-empty subset of $\mathbb{R}$ is connected $\iff$ interval

Thm 84. (subspace) $A \subset X$ connected $\implies$ $\overline{A}$ connected (*"closure of a connected space is connected"*)

Thm 85. ($f : X \to Y$) X connected $\implies$ $f(X) \subseteq Y$ is connected

Thm 1.26. ($f : X \to Y$) X compact $\implies$ $f(X) \subseteq Y$ is compact

Thm 87. X path-connected $\implies$ X connected

Thm 89. ($\iff$)Euclidean top/ $\mathbb{R}^n$: U connected $\iff$ U path-conn

Thm 100 / 104 / 126. Heine-Borel

An interval / subset of $\mathbb{R}$ / subset of $\mathbb{R}^n$ is compact $\iff$ closed and bounded

Thm 103. X compact, $Y \subseteq X$ closed: Y compact

Thm 105. X Hausdorff, $Y \subseteq X$ compact: Y closed (*counterexample for Hausdorff: pp top /* $\{x\}$)

Thm 108. X compact, $f : X \to \mathbb{R}$ cts: f is bounded and attains its bound

Thm 109. X compact, Y Hausdorff:

if $f : X \to Y$ is continuous and a bijection, then f^{-1} is also continuous

Thm 115. X non-empty, compact, Hausdorff, with no isolated points $\implies$ X uncountable

Lemma 122. Projections (product top.) are cts

Lemma 123. $f : X \to Y_1 \times Y_2$ is cts $\iff$ $p_1 \circ f$ and $p_2 \circ f$ are both cts

Thm 124 / 125. Tychonoff finite (bzw. arbitrary) products of compact spaces are compact (arbitrary only in Tychonoff topology, not box top)

Thm 127. X, Y connected $\implies$ $X \times Y$ connected

Thm 141. X compact and connected, $q : X \to X/\sim$: $X/\sim$ compact and connected

Thm 144. $q : X \to X/\sim$, $f : X/\sim \to Y$: f cts $\iff$ $f \circ q$ cts

Note Internet. X Hausdorff $\not\Longrightarrow$ $X/\sim$ Hausdorff (bug-eyed line)

Thm 166. X, Y path connected: $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$

Cor. 162. $\pi_1(X, x_1)$ is a group

Thm 165. π_1 is a top. invariant

Eg 168. $X \subseteq \mathbb{R}^n$ convex is simply connected

Lemma 173. Weak Lebesgue $X \subseteq \mathbb{R}^n$ compact with open cover $\mathcal{C}$: there is some $l > 0$ s.t. any subset of diameter $< l$ is contained in some $C \in \mathcal{C}$

Thm 174. Weak van Kampen $X = U \cup V$; U, V open, simply conn., $U \cap V$ path-conn: X is simply conn.

Lemma 182. Path-lifting : p a path in S^1 beginning at $(1, 0)$ has a unique $\tilde{p}$ in $\mathbb{R}$ beginning at $0 \in \mathbb{R}$, such that $c \circ \tilde{p} = p$

Lemma 183. Homotopy-lifting $F : [0, 1] \times [0, 1] \to S^1$ is a homotopy of paths with $F(0, s) = F(1, s) = (1, 0)$ there is a unique homotopy $\tilde{F} : [0, 1] \times [0, 1] \to \mathbb{R}$ with $c \circ \tilde{F} = F; \tilde{F}(0, s) = 0$ (for $s \in [0, 1]$)

Thm 184. Fundamental group of the circle $\pi_1(S^1) = \mathbb{Z}$

Cor. 185. $S^1 \ncong S^n (n > 1)$

Cor. 186. $\mathbb{T}^n$ hypertorus: $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$

Cor. 187. $\mathbb{T}^n \ncong \mathbb{T}^m (m \neq n)$

Thm 189. Universal covering space if X is path-conn, locally path-conn, semi-locally simply conn, "there must be a connected covering space that acts as a covering of all the other connected covers" = Universal covering space

Eg 198. Triangulate S^2: radial projection from e.g. tetrahedron

Page 77. Complexes are compact (by Heine-Borel as they are closed and bounded)

Lemma 211. Vertices of a simplex $\iff$ intersection of open stars is non-empty

Lemma 213. K an n-dimensional complex:

$$m(K^1) \leq \frac{n}{n+1} m(K)$$

Thm 214. Simplicial Approximation Theorem: there is one for K^n, n high enough

Page 86. Equivalence classes of edge loops form a group with operation traversing one path then the other: $E(K, v)$

Thm 225. $E(k, v) \cong \pi_1(|K|, v)$

Thm 226. $G(K, L) \cong E(K, v)$

where $G(K, L)$ for a complex K and path-conn, simply conn. subcomplex L with all vertices of K is defined by

- g_{ij} is a generator: $v_i v_j$ span a simplex
- $g_{ij} = e$: $v_i v_j$ span a simplex of the conn. subspace L
- $g_{ij} g_{jk} = g_{ij}$: $v_i v_j v_k$ span a simplex

Thm 232. Van Kampen *Only if we have to*

Lemma 234. $\partial_{q+1} \circ \partial_q = 0$

Page 94. $\texttt{im}(\partial_{q+1}) \trianglelefteq \texttt{ker}(\partial_q)$

Thm 236. First homology group $H_1(K) = \pi_1(K)/[\pi_1(K), \pi_1(K)$: *abelianisation of* π_1

Cor. 237. ($H_1 = \pi_1()$ if $\pi_1()$ abelian) a) $H_1(X) = 0 \Leftarrow X$ simply connected – b) $H_1(\mathbb{T}^2 = \mathbb{Z} \bigoplus \mathbb{Z}$ – c) $H_1(S^1) = \mathbb{Z}$
d) $H_1(KB) = \mathbb{Z}_2 \bigoplus \mathbb{Z}$

Lemma 238. $H_n(S^n) = \mathbb{Z}$; $H_m(S^n) = 0 (n > 0, m \neq n)$

Thm 239. For K, L complexes, any cts function $f : |K| \to |L|$ induces a group homomorphism $f_{n*} : H_n(|K|) \to H_n(|L|)$

Thm 230. For K, L, M complexes and f the identity map on $|K|$, each f_{n*} is the identity;
if $f : |K| \to |L|$ and $g : |L| \to |M|$ cts, $(g \circ f)_{n*} = g_{n*} \circ f_{n*}$

Thm 241. For K, L complexes and two homotopic functions $f, g : |K| \to |L|$:

$$f_{n*} = g_{n*} : H_n(K) \to H_n(L)$$

Cor. 242. $S^n \cong S^m \iff m = n$

Thm 243. Borsuk-Ulam If $g : S^n \to \mathbb{R}^n$ is cts, there is some $x \in S^n$ with $g(x) = g(-x)$

Cor. 244. For every $n \geq 0$, $S^n \ncong \subseteq \mathbb{R}^n$

Cor. 245. $R^m \cong R^n \iff m = n$

Cor. 246. Ham sandwich Given three closed convex sets $A_1, A_2, A_3 \subset \mathbb{R}^3$: there is a hyperplane of $\mathbb{R}^3$ that simultaneously bisects each of them

Lemma 249. $n \geq 2$: $\pi_n(X, x_0)$ is abelian

Lemma 250. X path-conn, $x_0, x_y \in X$: $\pi_n(X, x_0) \cong \pi_n(X, y_0)$

Lemma 251. X, Y path-conn: $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$