$f: X \to Y$

X Hausdorff $\Leftarrow$ Y Hausdorff

X path-conn $\Rightarrow$ Y path-conn

X conn $\Rightarrow$ ~~Y conn~~ f(X) conn

etc

X compact $\Rightarrow$ f(X) compact

Connected

- not disconnected
- no non-obvious clopen sets
- no "onto" functions to discrete top with $\{0, 1\}$
- union whose intersection is non-empty
- $\Leftrightarrow$ $\subseteq \mathbb{R}$: interval
- closure of a connected subspace
- image by cts f of a connected space
- path-connected
- $\Leftrightarrow$ $\subseteq \mathbb{R}^n$: path-connected
- quotient of connected space
- product of conn. spaces

Compact

- Every open cover has a finite subcover
- $\Leftrightarrow$ $\subseteq \mathbb{R}^n$: closed bounded — Heine-Borel
- Closed subset of a compact space
- image by cts f of compact space
- $\Leftrightarrow$ Intersection of all collections of closed sets with finite intersection prop. is non-empty
- quotient of compact space
- product of compact spaces

Compact $\Rightarrow$

Compact, f(X) cts: bounded, attains its bound

Compact subset of Hausdorff space: closed

Compact AND Hausdorff and no isolated points: UNCOUNTABLE!

Hausdorff

Hausdorff

Connected , path-connected

Compact

Cut-points , non-cut points

Fundamental group , π_n ← homotopy group for any n

H_n homology group

Useful homeomorphisms

- Stereographic projection : sphere − both poles (= plane − origin)

$$(x_1, \dots, x_{n+1}) \mapsto \left(\frac{x_1}{1-x_{n+1}}, \dots \frac{x_n}{1-x_{n+1}}\right)$$

sphere − one pole (= plane)

- $\tanh^{-1}$ or other asymptotic function : $\mathbb{R} \leftrightarrow (0,1)$

$c: \mathbb{R} \to S^1,\ x \mapsto e^{2\pi i x}$ view $S^1 \subset \mathbb{C}$

- Semicircle to line − just project x straight down

COVERING SPACE : $S^1 \subset \mathbb{C}$; $c_n : S^1 \to S^1,\ z \mapsto z^n$ // c_n is covering map for any n

$P_n = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = n^2\}$; $X_n = \bigcup_i P_i$

$c_n : X_n \to S^1,\ x \to \frac{x}{\|x\|}$

Distinguishing $[0,1]$, $[0,1)$, $(0,1)$: # non-cut points !

Analysis

- Monotone convergence
  bounded sequences converge

- Least upper bound
  bounded sets have a lub. not in $\mathbb{Q}$ if lub $= \sqrt{2}$

- Bolzano-Weierstrass
  bounded sequences have convergent subsequences not in $\mathbb{Q}$ if limit is irrational

- IVT
  See : connectedness

  but also: an infinite subset of a compact space must have a limit pt.

- A continuous function on a closed bounded interval is bounded and attains its bound.
  see: compactness

$f(x) = \frac{1}{x}$

$(0,1] \to \mathbb{R} \cong [0,1]$

(0,1] '' connected; [0,1] '' connected

Hausdorff: convergent sequences have unique limits

~~Limit points:~~

$f: X \to Y$

X discrete : f is continuous

Y indiscrete : f is continuous