Groups

Five groups of order 12:
A_4, C_{12}, $C_2 \times C_2 \times C_3$, $\cong C_2 \times C_6$
$Dih(12)$, Q_{12}

see ex66 for subgps.

123	–
213	(12)
231	(132)
321	(13)
312	(123)
132	(23)

subgroups

$\langle [\text{transposition}] \rangle$ $\quad a^2 = 1$ $\quad$ not normal $\quad \cong Z_2$

$\{1, (132), (123)\}$ $\quad : (132)^2 = (123)$ $\quad \cong Z_3$ $\quad = A_3$ (normal)

1234	–
2143	(12)(34)
2413	(1342)
4231	(14)
4321	(14)(23)
3412	(13)(24)
3142	(1243)
1324	~~(4)~~(23)
1342	(243)
3124	(123)
3214	(13)
2341	(1432)
2431	(1423)
4213	(134)
4123	(1234)
~~4132~~	(243)
1432	
1423	(234)
4132	(124)
4312	(1324)
3421	(1423)
3241	(1432)
2314	(132)
2134	(12)

Centre is trivial

(12) (13) (14) (23) (24) (34) $\quad$ 6
(12)(34), (13)(24), (14)(23) $\quad$ 3
(1342) (1432) (1423) (1243) (1234) (1324) $\quad$ 6

(243) (234) – 1
(143) (134) – 2
(123) (132) – 4
(124) (142) – 3

Subgroups:
$\cong S_3$: $\times 4$
A_4 normal
$\cong S_2$: $\times 6$ $\quad (Z_2)$
$Dih(8)$: $\langle (1234), (13) \rangle$: $(\times 3)$
$\langle (1234) \rangle$: Z_4

V_4 : $\{1, (12)(34), (13)(24), (14)(23)\}$
normal

$\langle (12), (34) \rangle$ (order 4)

double transp $\cong Z_2$

A_4

~~1234~~

(123) (132)
(124) (142)
(134) (143)
(234) (243)
(12)(34)
(13)(24)
(14)(23)
1

subgroups
$A_3 \cong Z_3$ $\quad$ not normal
V_4 normal
double transp. $\cong Z_2$ $\quad$ are not normal

centre is trivial

Dih(8)

vs Q_8 : $\langle a, b,\ a^4 = 1,\ b^2 = a^2,\ ba = a^{-1} b \rangle$

$\langle \alpha, \beta : \alpha^4 = 1, \beta^2 = 1, \beta\alpha = \alpha^{-1}\beta \rangle$

normal subgroup V_4 $\qquad Z(Dih(8)) = \langle \alpha^2 \rangle$
normal subgroup $\langle \alpha \rangle \cong Z_4$ $\qquad Z_2(Dih(8)) = D_8$
normal subgroup $\langle \alpha^2 \rangle \cong Z_2$

Dih(2n)

α a rotation
β some reflection

$\langle \alpha, \beta : \alpha^n = 1, \beta^2 = 1, \beta\alpha = \alpha^{-1}\beta \rangle$ $\qquad \{\alpha^i\beta : 0 \le i \le n-1\}$ are reflections

elements α^i, $\alpha^i\beta$ $\quad 0 \le i \le n$ $\qquad o(\alpha^i\beta) = 2$ for all i

$Z = \{1, \alpha^{n/2}\}$ (n even) or $\{1\}$ (n odd)

Dihedral Group

x^2

$\alpha^i \to \alpha^{2i} \to \alpha^{3i}$ order dividing n.

$(\alpha^i\beta) \to \cancel{\alpha}\ 1$ order 2

but $\{1, \alpha^i\beta\}$ is never normal

Subgroups order $4 \cong V_4$

$\{1, \alpha^{n/2}, \alpha^i\beta, \alpha^{i+n/2}\beta\}$

There are $n/2$ of these.

Subgroup order n (n even) normal

$\Rightarrow$ Dih(n) : $\langle\alpha^2, \beta\rangle$, $\langle\alpha^2, \alpha\beta\rangle$

gxg^{-1}

$\alpha^j\alpha^i(\alpha^j)^{-1} = \alpha^i$

$\alpha^j\beta\,\alpha^i(\alpha^j\beta)^{-1} = \alpha^{-i} \quad (\alpha^{2n-i})$

$\alpha^j(\beta\alpha^i\beta)\alpha^{-j} = \alpha^{2j+i}\beta$

$\alpha^j\beta(\alpha^i\beta)(\alpha^j\beta)^{-1} = \alpha^{2j-i}\beta$

$Z(G) = \{1, \alpha^{n/2}\}$ n even.

never use b^2 so identical

Quaternion Group Q_{4n}

$\langle a, b \mid a^{2n} = 1, b^2 = a^n, ba = a^{-1}$

x^2

$a^i \to a^{2i} \to$ order dividing $2n$

~~$(a^ib) \to a^{2i+n} \to a^{3i+n}b \to \dots$~~

~~order n dividing~~ $2n$

in particular

$b \to a^n \to a^nb \to 1$

$(a^ib) \to a^n \to a^{n+i}b \to 1$

Q_{12}: $\langle b\rangle$, $\langle ab\rangle$, $\langle a^2b\rangle$ have order 4
$\langle a\rangle$ has order 6 and is normal

gxg^{-1}

$a^ja^ia^{-j} = a^i$

$a^jba^i(a^jb)^{-1} = a^{-i}$

$a^j(a^ib)a^{-j} = a^{2j+i}b$

$a^jb(a^ib)(a^jb)^{-1} = a^{2j-i}b$

2-element group

$\{1, a^i\}$ is normal

$\iff a^i = a^{-i}$

$\iff (2n)$ is even (Q_{4n}) always

| n is even (D_{2n}) sometimes

$N \trianglelefteq G$

$K \trianglelefteq H \trianglelefteq$

then $KN \trianglelefteq HN$

$A \trianglelefteq G \trianglerighteq B$

$\left.\begin{matrix} A \cap B \\ AB \end{matrix}\right\} \trianglelefteq G$

$H \trianglelefteq G \trianglerighteq N$

$$\frac{H}{H \cap N} \cong \frac{HN}{N}$$

$H \trianglelefteq G$, $H \geq N \trianglelefteq$

$$\frac{G/N}{H/N} \cong \frac{G}{H}$$

$\trianglelefteq G/N$

$H \leq G$ soluble $\Rightarrow$ H soluble

$N \trianglelefteq G$ with $|G:N| = p$, $H \leq G$ $\Rightarrow$ $|H : N \cap H| = p$