Even/odd functions:

Even: $f(-x) = f(x)$
Odd $f(x) = -f(-x)$
$\Bigg\}$ for all x

of course, a function may be neither

even + even = even
odd + odd = odd
even * even = even
odd * odd = even
even * odd = odd
$\Bigg\}$ Ex 2.17

$$\int_{-m}^{m} f(x)\,dx = 2\int_0^m f(x)\,dx \quad \text{vs} \quad 0$$

$\Downarrow$ $f(x)$ even

$\Rightarrow f(x)$ odd

Fourier cosine $\rightarrow$ even function

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \; ; \; a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Fourier sine $\rightarrow$ odd function

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \; ; \; b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

non-periodic functions over a closed interval
$\Rightarrow$ even extension, or
$\Rightarrow$ odd extension

Fourier

- functions of period 2π are orthogonal f:

$$\int_{-\pi}^{\pi} f(x)g(x)\,dx = 0$$

$\sin(nx)\sin(mx)$
$\sin(nx)\cos(mx)$
$\sin(nx)\cos(nx)$
$\sin(nx)\cdot 1$
$1\cdot\cos(nx)$

ex 2.8

Thm: given f, g both periodic period: k, af+bg is also periodic k (any a, b)

- Euler functions period $2L$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$$

- Convergence of a periodic function
  - piecewise cts
  - at each point in the interval $[-L, L]$, the function has a LH and RH derivative:

$$\frac{f(x+\epsilon)-f(x)}{\epsilon} \qquad \lim_{\epsilon\to 0} \qquad \frac{f(x)-f(x-\epsilon)}{\epsilon} \qquad \text{exist}$$

PARTIAL FRACTIONS

- $(as+b)^n \rightarrow \dfrac{C_1}{(as+b)} + \cdots \dfrac{C_n}{(as+b)^n}$
- $(as^2+bs+c) \rightarrow \dfrac{Cs+D}{as^2+bs+c}$ (or n terms for repeated factors)
- Finding zeroes of polynomials : $\dfrac{\text{factors of } x^0 \text{ term}}{\text{factors of } x^n \text{ term}}$.
- Try setting (s = root) in the equation in the constants

CONVOLUTION

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

THEOREMS

① Existence

- ⇒ exponential order
- ⇒ piecewise continuous,
  bounded,
  finite # of discontinuities

② FIRST SHIFTING THM

$$\mathcal{L}(e^{at} f(t)) = F(s-a)$$

shifts s

③ PIECEWISE CTS WITH HEAVISIDE

$$\begin{cases} f_1 & t_1 \le t < t_2 \\ \vdots & \vdots \\ f_{n-1} & t_{n-1} \le t \le t_n \end{cases}$$

$$H(t-t_1)\cdot f_1 + H(t-t_2)(f_2 - f_1) + \dots + H(t-t_n)(\cancel{f_n} - f_{n-1}) = \$ - H(t-t_n)(f_{n-1})$$

④ SECOND SHIFTING THEOREM

$$\mathcal{L}(f(t-a)H(t-a)) = e^{-as}F(s)$$

shifts t

⑤ CHANGE OF SCALE

$$\mathcal{L}(f(\gamma t)) = \frac{1}{\gamma} F\left(\frac{s}{\gamma}\right)$$

⑥ DERIVATIVE OF A TRANSFORM

$$\frac{d^n F}{ds^n} = (-1)^n \mathcal{L}(t^n f(t)) \Rightarrow \boxed{\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n F}{ds^n}}$$

Common Laplace Transforms

	$f(t)$		$\mathcal{L}(f(t))$	
of a constant	a	$\Rightarrow$	a/s	$s > 0$
$f(t) =$	t		$\frac{1}{s^2}$	
more generally, $f(t) =$	t^n		$\frac{n!}{s^{n+1}}$	$\left(\frac{\Gamma(n+1)}{s^{n+1}} : n \notin \mathbb{Z}\right)$
	e^{at}		$\frac{1}{s-a}$	
	$\cos(at)$		$\frac{s}{s^2+a^2}$	$\frac{s}{(s^2-a^2)} = \cosh(at)$
	$\sin(at)$		$\frac{a}{(s^2+a^2)}$	$\frac{a}{(s^2-a^2)} = \sinh(at)$
of a derivative	$\cancel{\mathcal{L}}\ f'(t)$		$s\mathcal{L}(f(t)) - f(0)$	
	$\Rightarrow f^{(n)}(t)$		$\cancel{s^n \mathcal{L}(f(t)) \dots f^{(n-1)}(0)}$	
		$\rightarrow$	$s^n \mathcal{L}(f) - s^{(n-1)} f(0) - \dots \cancel{s} f^{(n-1)}(0)$	
of an integral	$\int_0^t f(\tau)\,d\tau$		$\frac{1}{s} F(s)$	
of a periodic function	$g(t)$, $0 \leq t \leq k$		$\frac{1}{1-e^{-ks}} G(s)$	NB: limits for g will not be $\to \infty$ so calculate integral.
using derivatives	$t^n f(t)$		$(-1)^n \frac{d^n F}{ds^n}$	if F known/easy for f
using integrals	$\frac{f(t)}{t}$		$\int_s^\infty F(\sigma)\,d\sigma$	if F known/easy and integrable
of a convolution	$(f * g)(t)$		$F(s)\,G(s)$	

Linear D.S.

$\dot{x}_i = a_{i1}x_1 + \dots + a_{ij}x_j + \dots$

simultaneous equations

$\Rightarrow \dot{\vec{x}} = A\vec{x}$

~~$\Rightarrow \ddot{x} + tr(A)\dot{x} + Det(A) = 0$~~ ~~solve as quadratic yields~~

$\Rightarrow \ddot{x} - \overset{a}{tr(A)}\dot{x} + \overset{b}{\det(A)}x = 0$

has standard solution as 2nd order ODE

$\Rightarrow$ solve for $x(t)$

$\Rightarrow$ sub. in to get $y(t)$

"aux. eqn"

$(m^2 - am + b) = 0$

$\Rightarrow$ two solns for m,

use boundary conditions $\Leftarrow$ $x(t) = Ae^{m_1 t} + Be^{m_2 t}$

PHASE PORTRAITS

vectors parallel to y

1) nullclines : $\dot{x} = 0$, $\dot{y} = 0$ $\leftarrow$ vectors parallel to x

2) fixed points $(\dot{x}, \dot{y}) = 0$ (intersection of nullclines)

$\dot{\vec{x}} = A\vec{x} \Rightarrow A\vec{x}_* = 0$

fixpoint

~~linear: fp @ 0~~
~~more fp~~

linear: fp @ 0
another fp $\Leftrightarrow \det A = 0$

solve as system of simultaneous equations

$\boxed{\lambda_1 < \lambda_2}$

then: characterise

$(A - \lambda I)\vec{v} = \vec{0} \Leftrightarrow \det(A - \lambda I) = 0$

for 2x2 matrix $\rightarrow$ quadratic in λ.

$\Downarrow$

$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$

from above $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow$ find $\lambda_1, \lambda_2 (, \dots)$

$\Rightarrow$ sub. in to obtain ~~$\vec{v}$~~ v_1, v_2

$\Downarrow$ in turn

$\Rightarrow$ eigenvectors : characterise per chart

$\rightarrow$ this works out to: $\boxed{\lambda^2 - tr(A)\cdot\lambda + \det A = 0}$

in $\boxed{\text{non-linear}}$ systems

– Hartman-Grobman:
linearise around fp using Jacobian:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

then evaluate at (each) fp.

Heat Equation : wire ends maintained at 0

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

where $\lambda_n = \frac{n\alpha\pi}{L}$

and $B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Heterogeneous: ends at T_1, T_2

$\Rightarrow$ Steady state $= \frac{(T_2 - T_1)}{L}x + T_1 = g(x)$

$\Rightarrow v(x,t) =$ homogeneous soln (ends at 0)

$\Downarrow$

$\Rightarrow u(x,t) = v(x,t) + g(x)$ | using $(f(x) - g(x))$ in B_n, not $f(x)$

Wave equation

constant for wave speed

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{cn\pi t}{L}\right)$$

(eg) length of string

where

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi x}{L}\right)dx$$

initial conditions $u(x,0) = f(x)$: initial position

$u_t(x,0) = g(x)$: initial velocity

$\Rightarrow 0$ if string is stretched then released

RICHARDSON

$$\frac{dx}{dt} = \beta y - \alpha x + \eta \qquad \Rightarrow A = \begin{pmatrix} -\alpha & \beta \\ \gamma & -\delta \end{pmatrix} \Rightarrow \text{find } \lambda \text{ \& ev}$$

$$\frac{dy}{dt} = \gamma x - \delta y + \theta$$

$$(x_*, y_*) = \left(\frac{\eta\delta + \theta\beta}{\alpha\delta - \gamma\beta}, \frac{\alpha\theta + \gamma\eta}{\alpha\delta - \gamma\beta} \right)$$

determinant
discriminant $\boxed{\alpha\delta - \beta\gamma}$

> 0 :	stable ~~arms race~~ spending
< 0 :	arms race

$\cancel{\beta}$

1. FP
2. λ ~~ev~~ (eigenvalues) (using A above)
3. ev $\Rightarrow$ phase portrait

$$\lambda = \frac{-(\alpha+\delta) \pm \sqrt{(\alpha+\delta)^2 - 4(\alpha\delta - \beta\gamma)}}{2}$$

by quad. form

$$\Downarrow$$

$$\begin{pmatrix} 2\beta \\ (\alpha - \delta) \pm \sqrt{(\alpha-\delta)^2 + 4\beta\gamma} \end{pmatrix}$$

LANCHESTER SQUARE

"aimed model"

$$\boxed{\frac{dx}{dt} = -\alpha y \qquad \frac{dy}{dt} = -\beta x}$$

linear: $fp = (0,0)$

$\lambda = \pm\sqrt{\alpha\beta}$, $ev = \begin{pmatrix} \alpha \\ \pm\sqrt{\alpha\beta} \end{pmatrix}$

$$\boxed{\alpha y^2 - \beta x^2 = c}$$

determinant

winner
$c > 0$: Y
$c < 0$: X
$c = 0$: –

LANCHESTER LINEAR

"unaimed model"

$$\boxed{\frac{dx}{dt} = -\alpha xy \qquad \frac{dy}{dt} = -\beta xy}$$

entire x and y axes are nullclines and fixed points

$c > 0$: y wins
$c < 0$: x wins

$$\boxed{\alpha y - \beta x = c}$$

$\alpha y \neq \beta x$

LANCHESTER GUERILLA

$$\boxed{2\alpha y - \beta x^2 = 2\alpha k}$$

$2\alpha k > 0$: y wins
$2\alpha k < 0$: x "

nb nullclines are parabolas!

$$\boxed{\frac{dx}{dt} = -\alpha y \;\;\text{(conventional)} \qquad \frac{dy}{dt} = -\beta xy \;\;\text{(guerillas)}}$$

Competing Species

sheep

goats

$$\frac{dx}{dt} = \alpha x - \beta x^2 - \gamma xy$$

$$\frac{dy}{dt} = \delta y - \eta y^2 - \theta xy$$

constants ≥ 0

FP: $(0,0)$

$(0, \delta/\theta)$

$(\alpha/\beta, 0)$

$$\frac{\theta\alpha - \delta\gamma}{\theta\beta - \eta\gamma}, \frac{\delta\beta - \alpha\eta}{\theta\beta - \gamma\eta}$$

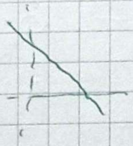
$\Downarrow$

exists if $\theta\beta - \eta\gamma \neq 0$

i.e. if the two lines intersect

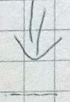
x-nullclines: $x = 0$

$y = \frac{1}{\gamma}(\alpha - \beta x)$



y-nullclines: $y = 0$

$y = \frac{1}{\eta}(\delta - \theta x)$



$$J = \begin{pmatrix} \alpha - 2\beta x_* - \gamma y_* & -\gamma x_* \\ -\eta y_* & \delta - 2\theta y_* - \eta x_* \end{pmatrix}$$

at $(0,0)$: $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \Rightarrow \alpha, \delta$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

unstable node

at $(0, \frac{\delta}{\theta})$: $-\delta, \; \alpha - \gamma\frac{\delta}{\theta}$

node type depends on sign of 2nd λ.

+ve: saddle

0: stable line

−ve: stable node

$(\frac{\alpha}{\beta}, 0)$: $-\alpha \quad \delta - \eta\frac{\alpha}{\beta}$

Logistic model

$$\frac{dP}{dt} = rP - \frac{r}{K}P^2$$

$$\begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \Rightarrow \lambda = x, z$$

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$$

$\boxed{\text{SIR}}$ — epidemics

$$\frac{dS}{dt} = -\beta S I$$

$$\frac{dI}{dt} = \beta S I - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

$$\frac{dI}{dS} = \left(\frac{dI}{dt} \Big/ \frac{dS}{dt}\right) = \frac{\gamma}{\beta S} - 1$$

$$\Rightarrow I = I_0 + S_0 - S + \boxed{\frac{\gamma}{\beta}} \ln\left(\frac{S}{S_0}\right)$$

"ρ" threshold value = corresp. to I_{max}

$= \frac{1}{R_0}$ reproductive rate

diagonal matrix

$\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$

$$\frac{dS}{dR} = (\dots) = \frac{\beta}{\gamma} S = -R_0 S$$

$$\Rightarrow S(t) = S_0 e^{-R(t) \cdot R_0}$$

Predator-prey = Lotka-Volterra

1) Fixed points
2) Linearise (J)
3) char. at each FP

rabbits $\quad \frac{dx}{dt} = \alpha x - \beta x y$

foxes $\quad \frac{dy}{dt} = \gamma x y - \delta y$

$(x_*, y_*) = (0,0)$ extinction

$= \left(\frac{\delta}{\gamma}, \frac{\alpha}{\beta}\right)$

x-nullclines: $x = 0$, $y = \alpha/\beta$

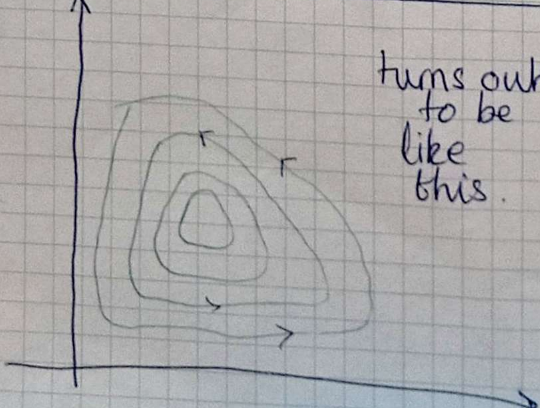
y-nullclines: $y = 0$, $y = \delta/\gamma$

FP: $0,0 \Rightarrow \lambda = \overset{\lambda_2}{\alpha}, \overset{\lambda_1}{-\delta}$

$ev = \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix}$ saddle point

$\frac{\delta}{\gamma}, \frac{\alpha}{\beta} \Rightarrow \pm i\sqrt{\alpha\beta}$ imaginary

this means we know nothing ⇒ can't use rules from linear systems here!



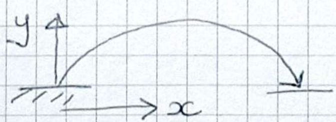
turns out to be like this.

Exp growth / decay:

$$\frac{dx}{dt} = \alpha x \Rightarrow x(t) = x_0 e^{\alpha t}$$

first order

Projectile

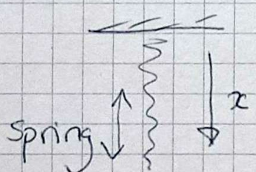


$$\frac{d^2x}{dt^2} = 0 \qquad \frac{d^2y}{dt^2} = -g$$

second order

Harmonic osc.

second order



$$\frac{d^2x}{dt^2} = -\omega^2 x$$

by c.o.v:

$$\frac{dx}{dt} = y \qquad \frac{dy}{dt} = -\omega^2 t$$

Change of variable: $y = \frac{dx}{dt}$.

Autonomous systems:

$$\frac{d_i x}{dt} = f_i(x_1(t), \dots x_n(t))$$

no explicit ~~∅~~ time-dependency

non-autonomous:
is it separable? $\frac{dx}{dt} = \cancel{f}\, g(x)h(t)$

$$\Rightarrow \int \frac{1}{g}\,dx = \int h\,dt$$

⇓
no
⇓

~~dum~~ dummy variable (after c.o.v → first order)

$\cancel{\frac{dz}{dt} \neq t}$

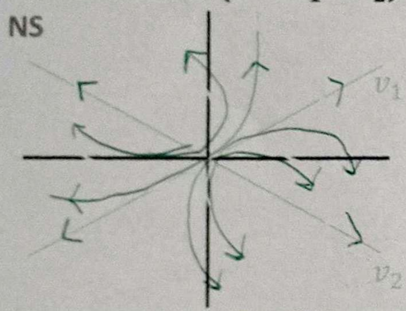
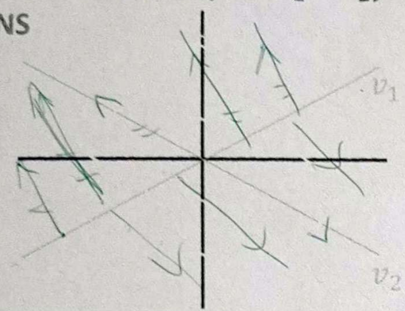
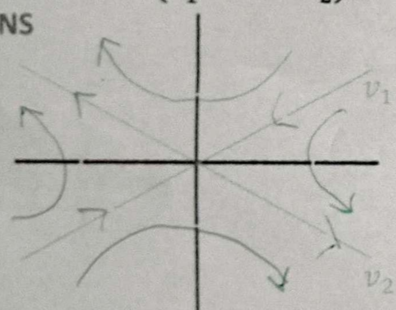
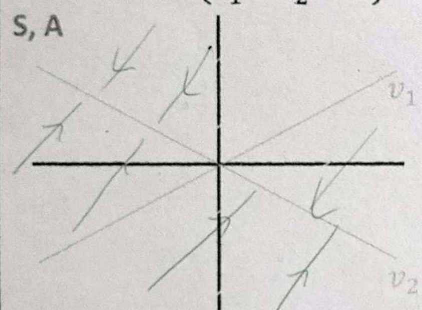
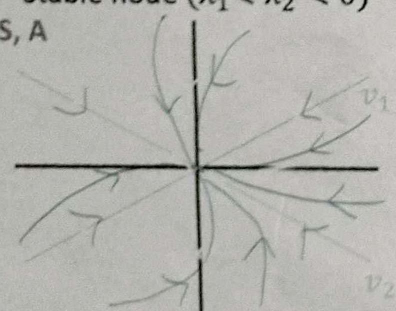
$z = t$

$$\frac{dz}{dt} = 1$$

Classification of the fixed points of 2D linear systems

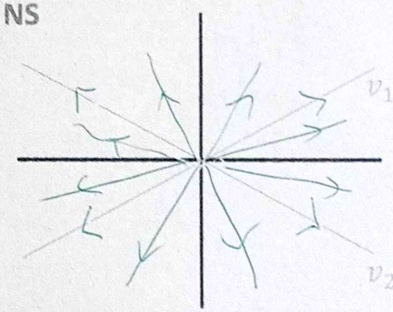
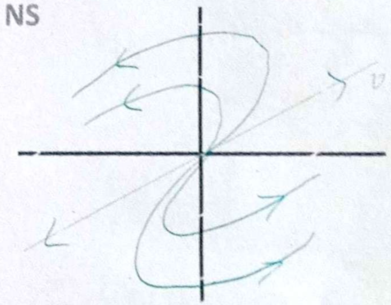
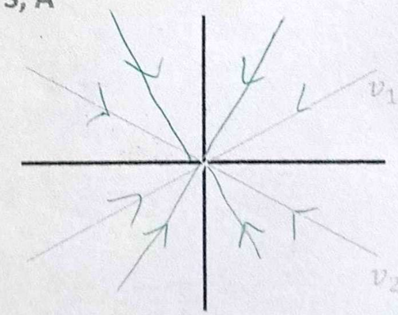
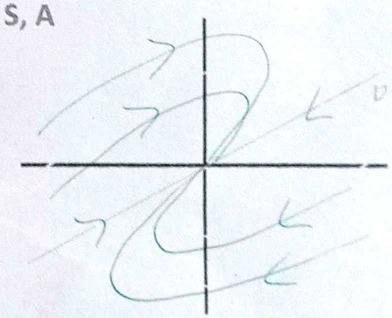
Two real distinct eigenvalues $(\lambda_1, \lambda_2 = \mathbb{R}, \lambda_1 \neq \lambda_2)$

Trajectory equation:
$\vec{x}(t) = c_1 e^{\lambda_1 t}\, \vec{v}_1 + c_2 e^{\lambda_2 t}\, \vec{v}_2$

Unstable node $(0 < \lambda_1 < \lambda_2)$ **NS** 	Unstable line $(0 = \lambda_1 < \lambda_2)$ **NS** 
Saddle $(\lambda_1 < 0 < \lambda_2)$ **NS** 	Stable line $(\lambda_1 < \lambda_2 = 0)$ **S, A** 
Stable node $(\lambda_1 < \lambda_2 < 0)$ **S, A** 	No motion $(\lambda_1 = \lambda_2 = 0)$ (Special, trivial case) All points $\vec{x}(t) \in \mathbb{R}^2$ are fixed points and so there is no motion anywhere.

Classification of the fixed points of 2D linear systems

Repeated eigenvalues $(\lambda_1, \lambda_2 = \mathbb{R}, \lambda_1 = \lambda_2 = \lambda)$

	Non-Degenerate matrix A (two distinct eigenvectors)	**Degenerate matrix A** (one eigenvector)
	Trajectory equation: $\vec{x}(t) = e^{\lambda t}(c_1\, \vec{v}_1 + c_2 \vec{v}_2)$	Trajectory equation: $\vec{x}(t) = e^{\lambda t}(c_1 \vec{v} + c_2(t\vec{v} + \vec{w}))$
$\lambda > 0$	Unstable star **NS** 	Unstable improper node **NS**  $\beta > 0 =$ clockwise rotation
$\lambda < 0$	Stable star **S, A** 	Stable improper node **S, A**  $\beta > 0 =$ clockwise rotation

Classification of the fixed points of 2D linear systems

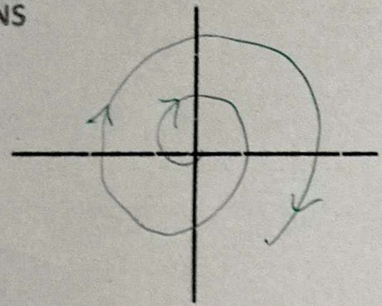
Complex eigenvalues ($\lambda_1, \lambda_2 = \mathbb{C}, \lambda_1 = \overline{\lambda_2}$)

Trajectory equation:

$$\vec{x}(t) = 2e^{at}(\alpha(\vec{u}\cos(bt) - \vec{w}\sin(bt)) - \beta(\vec{u}\sin(bt) - \vec{w}\cos(bt)))$$

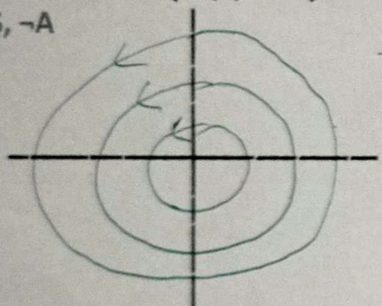
Unstable spiral ($\Re(\lambda) > 0$)

NS



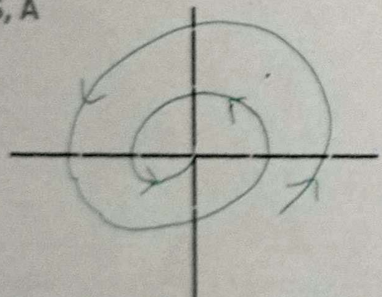
Centre ($\Re(\lambda) = 0$)

S, ¬A



Stable spiral ($\Re(\lambda) < 0$)

S, A



$\beta > 0$ = clockwise rotation