1 Notes

All subgroups of Dih(8): see eg. 1.17

2 Definitions

Group Require *Closure*, *Associativity*, *Identity*, *Inverses*

Abelian $xy = yx$ for all $x, y \in G$

Multiplication field $\mathbb{R}^*, \mathbb{C}^*, \mathbb{Q}^*$: no zero element → abelian group under multiplication

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Useful matrix groups GL_n : invertible ($\texttt{det} \neq 0$)

$$\begin{aligned} SL_n(F):&\quad \texttt{det} = 1 \\ SL_n^{\pm}:&\quad \texttt{det} = \pm 1 \\ UT_n:&\quad \text{upper triangular} \\ LT_n:&\quad \text{lower triangular} \\ Diag_n:&\quad \text{diagonal} \\ O_n:&\quad \{A : AA^T = I_n\} \end{aligned}$$

Symmetry group $\texttt{Sym}(\Phi)$: f is a symmetry if $f(\Phi) = \Phi$. Finite figures: rotations reflections; infinite figures: add translations, glide reflections

Regular polygon = $\texttt{Dih}(2n)$

Homomorphism $\theta(g_1 g_2) = \theta(g_1)\theta(g_2)$

Given a homomorphism, we know:

1. $\theta(1) = 1_H$
2. $\theta(g^{-1}) = \theta(g)^{-1}$
3. the order of $\theta(g)$ divides the order of g

Isomorphism Injective $f(x) = f(y) \implies x = y$

Surjective: for every y in H there is $x \in G$ with $y = f(x)$

NB: *Also require well-defined!*

Kernel Things in G that map to 1_H

NB: θ is injective if and only if $\texttt{ker}(\theta) = \{1_G\}$

Image Things in H that are $\theta(g)$ for some $g \in G$

Equivalence class Symmetric, reflexive, transitive. Elements *partition* the set. *Isomorphism* is an equivalence relation on the set of groups – *isomorphism classes*.

Index	of a subgroup H in a group G, $\|G : H\|$ is the number of distinct cosets of H making up G.
Conjugate	$y\, \texttt{conj}\, x$ if $y = gxg^{-1}$ for some g. Sometimes write x^g for gxg^{-1}
Conjugacy class	$x^G = \{gxg^{-1} : g \in G\}$
Centraliser	$C_G(x)$ all g that commute with x
Centre	$Z(G)$ all g that commute with *every* x.
Cycle type	of a permutation in S_n: number of cycles of each length (ignoring 1)
Normaliser	of a subgroup: $N_G(H) = \{g : gHg^{-1} = H\}$
Normal subgroup	$gHg^{-1} = H$ for all g.
Quotient	Of a group G by a normal subgroup N is the set of cosets of N (generally we say left cosets, but L and R are equivalent here). Group multiplication is defined as $(gN)(hN) = (gh)N$.
Generated subgroup	$\langle H, K\rangle$ of two groups – it's the group they generate...
Product	of two groups H, K: $HK = \{hk : h \in H, k \in K\}$. This is a *set*, not a subgroup. We might get lucky and find it's a group...
Acts on	A group G acts on a set X (a *G-set*) with some operation $\cdot$ if: 1. for all $x \in X$, $1 \cdot x = x$ 2. for all $x \in X$, $g_1, g_2 \in G$, $(g_1g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$
Orbit	Given $x \in X$, a G-set (acted on by G, take $orb(x) = \{g \cdot x : g \in G\}$. ("Where can x get sent?"). The size of this set is the *length* of the orbit. (*A subset of* X)
Stabiliser	of x, G_x is the set of g that don't change x: $\{g \in G : g \cdot x = x\}$
Sylow p-subgroup	Has order p^n, where $\|G\| = p^n m$ (i.e. not p^r, $r < n$: this is a general $p-subgroup$
Simple	A group is simple if it has no non-trivial proper normal subgroups.
Direct Product	Pointwise ordered pairs (h, k)
Internal direct product	If $G = HK$, then G is the *internal direct product* of H, K
Elementary abelian	Abelian group whose elements all have the same order. By Cauchy this must be a prime p.

Type	of a finite abelian p-group: $G \cong C_{p^{\lambda_1}} \times \ldots C_{p^{\lambda_r}}$: type is $[\lambda_1, \ldots, \lambda_r]$, sorted largest to smallest (can do this as the group is abelian)
Upper central series	1. Let $Z_1(G) = Z(G)$ 2. Let $Z_i(G)$ be the group such that $Z_i(G)/Z_{i-1}(G) = Z(G/Z_{i-1}(G))$ 3. $\{1\} \leq Z_1(G) \leq$ ... is the upper central series
Nilpotent	If $Z_n(G) = G$ for some n, G is nilpotent, *of class* n.
Subnormal series	Chain of subgroups where each one is normal in the one above. *Subnormal refinement* to *composition series* (all subnormal refinements have repeating terms).
Normal series	Subnormal series where each one is ALSO normal in the main one. *Normal refinement* to *chief series*.
Isomorphic subnormal	series if they have isomorphic factors *including repetitions*.
Composition factors	Quotients of a composition series.
Chief factors	Quotients of a chief series.
Automorphism	An automorphism of a group G is an isomorphism from G to G.
Characteristic subgroup	N is a characteristic subgroup of G if $f(N) = N$ for all automorphisms of G. *Since conjugation by any element of G is an automorphism, N must be normal!*
Characteristically simple	G is char. simple if its only characteristic subgroups are $\{1\}$ and G.
Soluble	A group is soluble if it has a normal series whose factors are all abelian. *This is NOT required to be a chief series so an abelian group $G > \{1\}$ and done.*
Commutator	the commutator $[g, h]$ of $g, h \in G$: $[g, h] = ghg^{-1}h^{-1}$
Commutator subgroup = derived group	G' is the subgroup generated by the commutators of G: $$G' = [G, G] = \langle ghg^{-1}h^{-1} : g, h \in G \rangle$$
Derived series	of a group G is the series of commutator subgroups: $$G^{(0)} = G; G^{(1)} = [G, G]; \ldots ; G^{(r)} = [G^{(r-1)}, G^{(r-1)}]$$

3 Theorems and results

Prop 1.4. Identity is unique: gh = g ⇒ h = I_g – inverses are unique – $gh = 1 \Rightarrow h = g^{-1}$

$x^2 = 1 \forall x \in G : G$ is abelian one way

Page 5. $(x_1 x_2 \dots x_k) = (x_1 x_k)(x_1 x_{k-1}) \dots (x_1 x_2)$ (For product of transpositions)

Prop 1.7. [Subgroup test] Test closure, inverses suffices

Prop 1.8. [Finite subgroup test] Test closure suffices

Prop 1.13. Let G be a group with $x \in G$: $\langle x \rangle = \{x^k : k \in \mathbb{Z}\}$ is a subgroup of G with $|\langle x \rangle = o(x)|$. Eg in $\mathbb{C}$: $\langle i \rangle = \{i, -1, -i, 1\}$ (not everything has infinite order!)

Thm 1.15. Lagrange $H \leq G$: $|H|$ divides $|G|$.

Cor. 1.15. (1) G has prime order $\implies$ G is cyclic (2) G has prime order $\implies$ any non-identity elt is a generator (3) Order of any element in the group divides $|G|$ (4) $a \in G : a^{|G|} = e$ (5) Every subgroup of a cyclic group is cyclic

Note p.7. To find subgroups systematically, look at groups generated by subsets

Page 9. θ is injective if and only if $\texttt{ker}(\theta) = \{1_G\}$

Prop 1.21. Kernel, image are groups. $\texttt{ker}(\theta) \leq G; \texttt{im}(\theta) \leq H$ (Ex. 22: $\texttt{ker}(\theta)$ is normal in G)

Lemma 1.23. Element orders are preserved by isomorphism. $f : G \to H$ isomorphism: $o(g) = o(f(g))$.

Lemma 1.24. Properties preserved by isomorphism: (1) Group order – (2) Abelianism – (3) Cyclicness (4) Number of elements of each order

Page 11. $\mathbb{Z}_n = C_n$

Lemma 1.30. $g_1 \sim g_2 \iff g_1 \in Hg_2$ is an equivalence relation with cosets as equivalence classes (so the cosets partition the group).

Lemma 1.31. Equality of cosets: $H \leq G$: $g_1 H = g_2 H \iff g_1^{-1} g_2 \in H$; similarly $Hg_1 = Hg_2 \iff g_1 g_2^{-1} \in H$.

Page 12. $\dfrac{|G|}{|H|} = |G : H|$

Page 12. Conjugacy is an equivalence relation, i.e. conjugacy classes partition the group.

Ex 19. Centre / centraliser $Z(G)$ and $C_G(x)$ are subgroups of G.

Prop 1.33. Conjugacy class x^G vs centraliser: $|G| = |x^G| \cdot |C_G(x)|$.

Lemma 1.34. If y is conjugate to x, x and y have the same order.

Lemma 1.35. Conjugates in permutation group S_n: Given $x, g \in G$: gxg^{-1} is obtained by replacing each i in x with $g(i)$.

Note internet. Inverse of a permutation in cycle notation: write it backwards.

Thm 1.36. Permutations in S_n are conjugate if and only if they have the same cycle type.

Ex 20. Any subgroup of an abelian group is normal.

Ex 21. The centre $Z(G)$ of any group G is normal in G.

Ex 22. Kernel of any homomorphism from G to another group is normal in G.

Lemma 1.37. Intersection of a normal subgroup with any subgroup H is a normal subgroup of H (but not necessarily of the original group G): $N \trianglelefteq G, H \leq G : N \cap H \trianglelefteq H$

Lemma 1.38. Index 2 $\implies$ H is normal in G.

Ex 23. A subgroup H is normal in G if and only if H is a union of conjugacy classes of G.

Ex 24. List of conjugacy classes of S_4 and normal subgroups of S_4; normal subgroups of A_4.

Thm 1.40. *The homomorphism theorem* $\dfrac{G}{\texttt{ker}(\theta} \cong \texttt{im}(\theta)$.

Prop 1.41. The product HK is a subgroup a group G if and only if $HK = KH$.

Prop 1.42. If N is a normal subgroup of G and H any subgroup of G, $NH = HN$ so NH is a group.

Cor. 1.43. For an abelian group G, HK is always a subgroup.

Lemma 1.44. $|HK| = \dfrac{|H| \cdot |K|}{|H \cap K}$ $|G| \geq |HK|$, even if HK is not a subgroup

Prop 2.5. G is a Gset (acting on itself) under the actions <u>left-multiplication</u>: $g \cdot x = gx$, <u>inverse right multiplication</u>: $g \cdot x = xg^{-1}$, conjugation $g \cdot x = gxg^{-1}$. *But not by regular right multiplication,* unless G is abelian

Eg 2.7. G a group, X the set of subgroups of G: G acts on X by conjugation.

Eg 2.8. G a group, $H \leq G$, G acts by left mult. on the left cosets of H, and right inverse mult. on the right cosets of H.

Prop 2.10. Orbits are equivalence classes: $x\ y$: $x = g \cdot y$ for some g is an equivalence relation.

Cor. 2.11. ... and therefore orbits *partition* X

Prop 2.12. The stabiliser of x, G_x, is a subgroup of G.

Thm 2.13: Orbit-stabiliser. $|orb(x)| = |G : G_x| \implies$
$|orb(x)| = \dfrac{|G|}{|G_x|}$. *Prove with a bijection: check* well-defined, injective, surjective

Eg 2.15. Let G act on itself by conjugation: the orbits are the conjugacy classes:

$$|x^G| \times |C_G(x)| = |G|$$

(stabiliser-centraliser): this tells us that *The size of a conjugacy class divides the order of the group*

Eg 2.16. Let G act on its subgroups by conjugation. Orbit of a subgroup is its conjugates. Stabiliser of a subgroup is its normaliser: *Number of conjugates of H is the index of $N_G(H)$ in G*

Prop 2.17. p prime, order of G is p^n for some $n \in \mathbb{Z}^+$: The centre of G ($Z(G)$) is non-trivial. *Proof via orbit-stabiliser*

Page 32. Corollary: no simple groups of order p^n

Prop 2.18. If G is a group such that the quotient by its centre is cyclic, i.e. $G/Z(G)$ is cyclic, G is abelian.

Prop 2.19. If G has order p^2 for some prime p, G is abelian.

Thm 2.20. Sylow. $|G| = p^n m; p, m$ coprime, $n > 1$

Let n_p be the number of Sylow p-subgroups.

- G contains at least one Sylow p-subgroup
- $n_p \equiv 1 (modp)$; n_p divides $|G|$
- If $Q \leq G$ and $|Q| = p^r$, Q is contained in some Sylow p-subgroup:
- Sylow p-subgroups form a single conjugacy class

Tech. lemma 2.22. p prime, $m \in \mathbb{Z}^+$, m, p coprime: $\begin{pmatrix} p^n m \\ p^n \end{pmatrix} = m(modp)$

Lemma 2.24. $|G| = p^n m$. G has at least one Sylow p-subgroup.

Lemma 2.25. Number of Sylow p-subgroups, $n_p \equiv 1 (modp)$.

Prop 2.26. For P a Sylow p-subgroup, and $Q \leq N_G(P)$: Q is contained in P. *(Use Lemma 1.44).*

Lemma 2.28. Given G with $|G| = p^n m$ and P a Sylow p-subgroup of G, $Q \leq G$ with $|Q| = p^r (r \geq 0)$: Q is contained in some conjugate of P. *Let Y be the conjugacy class of P; let G act on Y by conjugation.*

Lemma 2.29. Sylow p-subgroups form a single conjugacy class of subgroups, so n_p divides $|G|$. (*see Eg. 2.15...*)

Ex 36. G has order pq product of two primes, $q \neq 1 (modp)$: G is cyclic.

Thm 2.34. Cauchy. If p prime divides the order $|G|$ of G, then G has at least one element of order p. *(If $g \in P$ a Sylow p-subgroup has order p^k, just take $g^{p^{k-1}}$)*

Lemma 2.35. Group of order 4 is either cyclic or V_4.

Thm 2.36. If $|G| = 2p$ for p an odd prime, G is either cyclic or dihedral

Page 31. Simple groups are: Cyclic of prime order – A_n $(n \geq 5)$ – "of Lie type" – Sporadic Simple (there are 26 of these)

Prop 2.37. G abelian simple (finite, non-trivial): G is cyclic of prime order

Ex 41. No simple groups of order pq for primes p, q.

Ex 42. For any even integer $n > 2$, there are AT LEAST two non-isomorphic groups: C_n and $\texttt{Dih}(n)$

Page 32. Example of a group of Lie type.

Page 33. $|GL_2(\mathbb{Z}_p)| = (p^2 - 1)(p^2 - p)$

Page 33. $n \times n$ matrix over $\mathbb{Z}_p$ is invertible if and only if rows are linearly independent.

Ex 45. $Z(SL_2(\mathbb{Z}_p)) = \{\pm I\}$

Ex 46. Being in $GL_n(\mathbb{Z}_p)$ means having linearly independent rows. There are p^n possible first rows, but they cannot all be zero or we would get det zero. So $p^n - 1$ possible first rows.
Next row cannot be an integer multiple of first row, and in $\mathbb{Z}_p$ there are p possible integer multiples. So $p^n - p$ possibilities.

So there are $\prod_r (p^n - p^r)$ possible rows altogether for an $r \times r$ matrix in GL_n over $\mathbb{Z}_p$.
$|SL_n(\mathbb{Z}_{p)}| = \dfrac{|GL_n(\mathbb{Z}_p)|}{|\mathbb{Z}_p|}$
$SL_3(\mathbb{Z}_2) = GL_3(\mathbb{Z}_2)$.

Ex 47. $|G| = 2pq$ then G has a normal subgroup either P or Q (Sylow subgroups) and furthermore a normal cyclic subgroup PQ.

Ex 48. $|G| = mp^n$ where $m < p$: G cannot be simple (since $n_p \equiv 1 (mod p)$ but $m < p$ forces $n_p = 1$).

Ex 49. Summary:

- $|G| = p^n (n > 1)$: G is not simple
- $|G| = mp^n (m < p)$: G is not simple (get additional reqs)
  (NB this covers the case $G = pq$: just set $m = p, p = q, n = 1$).
- $|G| = 2pq$: G is not simple

Lemma 2.40. $n \geq 3$: A_n is generated by 3-cycles.

Thm 2.41. $n \geq 5$: A_n is simple.

Ex 51. $H \times K$ (with pointwise multiplication) is a group

Eg 3.2. Is the direct product of two cyclic groups cyclic? Answer: generally, no. $G = C_r \times C_s$ is cyclic if and only if $gcd(r, s) = 1$

Page 38. Order of $H \times K$ is $|H| \cdot |K|$.

Ex 53. $\hat{H} = \{(h, 1) : h \in H\}$ is a normal subgroup of $H \times K$ and $\hat{H} \cong H$; *Normally write as if* $H \leq H \times K$.

Lemma 3.3. If $H, K \trianglelefteq G$ (H, K are NORMAL subgroups) and $H \cap K = \{1\}$, then $\langle H, K\rangle = HK \cong H \times K$.

If $|H| \times |K| = |G|$, then $H \times K \cong G$. *Prove isomorphism by exhibiting an isomorphism!*

Prop 3.4. G finite, H_i normal subgroups of G, $H_i \cap H_j = \{1\}$ for all $i \neq j$, $|G| = \sum |H_i|$: then

$$G \cong H_1 \times \dots H_r$$

Lemma 3.5. Given normal $H, K \trianglelefteq G$ and $H \cap K = \{1\}$, if every element of G is hk for some h, k, then $G \cong H \times K$

Ex 55. If $G = HK = H \times K$, then every element of G has a unique form hk.

Thm 3.6. Every finite nontrivial abelian group is the internal direct product of its Sylow subgroups.

Prop 3.7. G abelian and $G \cong H_1 \times ... \times H_r$: Let p be a prime dividing $|G|$ and P_i be the Sylow p-subgroup of H_i: then the Sylow p-subgroup of $G \cong P_1 \times \dots \times P_r$.

Ex 56. $H \times K \cong K \times H$ (any H, K)

Prop 3.8. A group of order p^2 is either cyclic or isomorphic to $C_p \times C_p$. (*We already know it is abelian!*)

Tech. lemma 3.9. s, n positive integers, $s \leq n$, p prime. The set of elements of order dividing p^s in $C_{p^n} = \{g^{dp^{n-s}} : d \in \mathbb{Z}\}$.

Lemma 3.10. G abelian p-group, a an element of maximal order, $H = \langle a\rangle$. bH an element of G/H with order p^m: then bH contains an element of order p^m in G. (*Not true in general*)

Thm 3.11. G abelian p-group: is an internal direct product of cyclic p-groups.

Thm 3.12. G a finite abelian p-group: every decomposition of G as a direct product of cyclic p-groups has the same type.

Thm 3.13. Every finite nontrivial abelian group is isomorphic to a direct product of cyclic p-groups; Decomposition is unique up to the order of the factors.

Lemma 3.16. $G \cong C_{n_1} \times C_{n_2} \times \dots$: then $G \cong C_{n_1 n_2} \dots$ if and only if n_1, n_2 (etc) are pairwise coprime.

Ex 62. Number of non-isomorphic abelian groups corresponds to integer partitions of some n

Eg 3.19. All groups order 4 are abelian

Prop 3.21. G a group with normal subgroup $N \trianglelefteq G$; furthermore $Q \leq G/N$ is a subgroup of the quotient of G by N. Then: $H = \{g : gN \in Q\}$ is a subgroup of G; $N \leq H$; $H/N = Q$; Q is normal in G/N if and only if H is normal in G.

Eg 3.22. Abelian group is nilpotent of class 1

Lemma 3.24. Every finite p-group is nilpotent.

Thm 3.25. If G is an internal direct product of its Sylow subgroups, G is nilpotent. Conversely, if G is nilpotent, G is an internal direct product of its Sylow subgroups.

Page 49. : Reminder: $(gN)^k = g^k N$

Page 50. : Summary of always-normal subgroups:

- $\{1\}, G, Z(G)$
- Kernel of any homomorphism
- $N \cap H$ is normal in H (assuming N is normal in G)
- Subgroup of index 2
- Union of conjugacy classes (IF and ONLY IF)
- if N is normal, then $NH = \langle N, H \rangle$

Ex 66. There are only two non-isomorphic abelian groups of order 12

Thm 4.2. $H \leq G$, $N \trianglelefteq G$:

$$\frac{H}{N \cap H} \cong \frac{HN}{N}$$

Ex 69. N a normal subgroup of prime index p: $\dfrac{|G|}{|N|} = p$. $H \leq G$ not contained in N: $N \cap H$ has index p in H.

Thm 4.3. H, N BOTH normal subgroups of G with $N \leq H$. Then H/N is a normal subgroup of G/N, with

$$\frac{G/N}{H/N} \cong \frac{G}{H}$$

Ex 4.9. Any *nilpotent* group G has upper central series and this (reversed) is a normal series for G. *Not an if and only if: non-nilpotent groups can still be soluble...*

Ex 4.11. In an *abelian* group, any subnormal series is also a normal series.

Ex 82. Group order is a power of 2: $|G| = 2^n$: G has a subgroup of index 2.

Ex 83. The only normal subgroups of $S_n, n \geq 5$ are A_n and the two trivial subgroups. (*Use N normal in S_n implies $N \cap A_n$ is normal in A_n*).

Lemma 4.15. Zassenhaus. . Let H and K be subgroups of G.
Let $A \trianglelefteq H$, $B \trianglelefteq K$:

$$\frac{(H \cap K)A}{(H \cap B)A} \cong \frac{(H \cap K)B}{(A \cap K)B}$$

Thm 4.16. Schreier's refinement theorem. . Given S, T subnormal (resp. normal) series for G, there exist S', T' subnormal (resp. normal) refinements of S, T such that S' and T' are isomorphic. (They might be trivial refinements).

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Lemma 4.17. N, H, K subgroups of G, and $K \leq H$. If N is normal in G and K is normal in H, i.e. if $N \trianglelefteq G, K \trianglelefteq H, H \leq G$, then:
KN is normal in HN.

Ex 86. A, B normal subgroups of G: then $A \cap B$ is normal in G and AB is normal in G.

Thm 4.18. Jordan-Hölder. If a group has a composition series (chief series) (it might not – if it's infinite!) then any two composition series (chief series) for that group are isomorphic.

Prop 4.21. If a is a composition factor of a group, A is simple.

Cor. 4.22. The only composition factors of nontrivial finite abelian groups are cyclic of prime order.

Ex 88. All abelian simple groups are finite.

Ex 89. An infinite abelian group has no composition series.

Thm 4.23. Fundamental thm of arithmetic. . Every integer greater than 1 can be uniquely factorised as a product of prime numbers.

Page 63. All characteristic subgroups are normal (but not necessarily vice versa...)

Page 64. If there is just one subgroup of order n, that subgroup must be characteristic.

Prop 4.29. . If G is a finite, characteristically simple group, then: G is an internal direct product of isomorphic simple groups.

Eg 4.30. The only simple group that can be a subgroup of a group of order 8 is C_2.

Eg 4.31. The nontrivial finite abelian characteristically simple groups are the elementary abelian p-groups.

Thm 4.32. A chief factor of a group is characteristically simple, and hence is a direct product of isomorphic simple groups.

Eg 4.33. Chief factors of A_4 are C_3 and V_4.

Cor. 4.34. A nontrivial abelian chief factor of a finite group is an elementary abelian p-group.

Thm 4.35. The smallest nonabelian simple group is A_5

Cor. 4.36. If G has order > 1 and less than 60, its composition factors are cyclic of prime order and its chief factors are elementary abelian.

Eg 4.39. All abelian groups are soluble.

Lemma 4.40. A finite group G is soluble if and only if the chief factors of G are abelian, i.e. if and only if the chief factors are elementary abelian.

Eg 4.41. All groups of order less than 60 are soluble.

Prop 4.45. If G is soluble and H is a subgroup of G, then H is soluble.

Prop 4.46. If G is a finite, simple, soluble group then G is cyclic of prime order.

Page 68. . If g, h commute, $[g, h] = 1$. In fact, G abelian if and only if $G' = 1$.

Ex 98. . $[g, h] = [h, g]$.

Thm 4.49. G a group. The derived group G' is a characteristic subgroup of G.
Moreover, the derived group G' is the smallest normal subgroup with abelian quotient. (ie if G/N is abelian, $G' \leq N$).

Thm 4.52. G is soluble if and only if $G^{(r)} = \{1\}$ for some r.

Prop 4.54. There are two non-isomorphic non-abelian groups of order 8: $\mathtt{Dih}(8)$ and Q_8.

Prop 4.56. There are three non-isomorphic non-abelian groups of order 12: $\mathtt{Dih}(12), Q_{12}, A_4$.

Exam 2015. Every quotient of a finite cyclic group is itself a finite cyclic group. Quotient of an abelian group is abelian.