1 Sn

- Inverses: reverse
- Conjugate x by g: (=gxg^{-1}) replace each i in x with $g(i)$
- Permutations are conjugate $\iff$ same cycle type
- A_n simple $n \geq 5$
- A_n generated by $3 - cycles$ $(n \geq 3)$
- A_n is the only nontrivial proper subgroup of S_n, $n \geq 5$
- $(x_1 x_2 \ldots x_k) = (x_1 x_k)(x_1 x_{k-1}) \ldots (x_1 x_2)$

2 p-groups

Type (for abelian): counts of powers of p $G \cong C_{p^{\lambda_1}} \times \ldots C_{p^{\lambda_r}}$: type is $[\lambda_1, \ldots, \lambda_r]$

- $|G| = p^2 \implies G$ abelian
- $Z(G)$ is nontrivial (but could be whole group)
  $\implies$ never simple!
- If abelian p-group: idp of cyclic p-groups
- Abelian p-group: every decomposition as a direct product of cyclic p-groups has same type
- Finite p-group: nilpotent

3 Abelian groups

- Elementary: all elements have same order
- nilpotent of class one
- factors of a soluble group
- $x^2 = 1 \forall x \implies$ abelian
- All subgroups are normal; $\implies HK$ is always a subgroup
- $G/Z(G)$ cyclic $\implies G$ was abelian
- $|G| = p^2 \implies G$ abelian
- Abelian and simple : cyclic of prime order
- If abelian p-group: idp of cyclic p-groups

- Abelian p-group: every decomposition as a direct product of cyclic p-groups has same type
- Number of non-isomorphic abelian groups corresponds to integer partitions of some n
- All (two!) groups order 4 are abelian

4 Any group

- Any finite char. simple group is an IDP of isomorphic simple groups
- Any finite simple soluble group is cyclic of prime order
- Nilpotent $\iff$ IDP of sylow subgroups

4.1 Order

- Conjugate elements have the same order
- $HK = \frac{|H|.|K|}{|H \cap K}$ (Whether or not HK is a group!)
- $|G| = ...$

$$\begin{aligned} p^n &\implies Z > 1 \\ p^2 &\implies G \text{ abelian} \\ &\implies \cong C_p \times C_p \text{ or } \cong C_{p^2} \\ 2p &\implies G \text{ cyclic or dihedral } (p \text{ odd}) \\ 2pq &\implies \text{exists } P, Q \trianglelefteq G; PQ \cong C_{pq} \\ pq &\implies \text{exists } P, Q \trianglelefteq G \\ pq \text{ with } q \neq 1(modp) &\implies G \text{ cyclic} \\ 4 &\implies G \cong C_4 \text{ or } G \cong V_4 \\ n > 2 &\implies C_n \ncong \texttt{Dih}(n) \end{aligned}$$

5 Normal groups // Simple groups

5.1 Conjugacy

- Conjugacy classes partition a group
- Conjugate elements have the same order // in S_n same cycle type
- $|G| = |x^G| \cdot |C_G(x)|$
- $\Rightarrow$ Conjugacy class size divides order of group
- Conjugate in $H \leq G \implies$ conjugate in G (one-way)

5.2 General

- Normal $\iff$ union of conjugacy classes
- Sylow p-subgroups form a single conjugacy class
- Number of conjugates of a group is the index of $N_G(H)$ in the group
- $\texttt{ker}(\theta)$ – $Z(G)$ – $N \cap H \trianglelefteq H$ – index 2 – union of conj. classes
- if $H \leq G, N \trianglelefteq G$, then $HN = NH$
- if $HN = NH$, then the product HN is a group
- $|G| = pq$: G cannot be simple
- $|G| = 2pq$: G has a normal subgroup P or Q and a normal cyclic subgroup PQ
- $|G| = mp^n, m < p$: G cannot be simple (ex 48)
- $|G| = p^n (n > 1)$: G is not simple
- $\{(h, 1) : h \in H\} \trianglelefteq H \times K$
- $H, K \trianglelefteq G, H \cap K = \{1\} \implies \langle H, K\rangle = HK$
-
$$\frac{G/N}{H/N} \cong \frac{G}{H}$$
(require $N, H \trianglelefteq G$)
- $|G| = 2^n$: G is not simple (has a subgroup index 2)
- Compo factors are simple